Higher Spin Quantum Fields as Twisted Dirac Fields

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Abstract. We pursue the idea of constructing higher spin fields as solutions to twisted Dirac operators. As general results we find that twisted prenormally hyperbolic first order operators (such as the Dirac operator) on globally hyperbolic Lorentzian spacetime manifolds have unique advanced and retarded Green's functions and their Cauchy problem is well-posed. The space of compactly supported Cauchy data can be equipped with a naturally induced Hermitian but generally indefinite product, which is shown to be independent of the underlying choice of Cauchy hypersurface.

As a special result, we study one particular example of twisted Dirac operator for Fermionic fields of higher spin. It allows a canonical Hermitian product on the spinor bundle, but the resulting product on the space of compactly supported Cauchy data fails positivity. Thus, it is shown that this construction does not allow quantization by forming a C^*-algebra representation of the canonical anti-commutation relations in the well known fashion generalizing the spin $\frac{1}{2}$ construction by Dimock (1982).

This document is equipped with an introduction to the formalism of 2-spinors on curved spacetimes in an invariant fashion using abstract index notation.

CONTENTS

I. INTRODUCTION AND MOTIVATION

In the setting of general relativity, various types of physical fields (and, in a suitable sense, also quantized fields) can be described as solutions to a differential equation

$$\mathcal{T}\Phi + \boldsymbol{i}\, m\, \Phi = 0\,, \qquad \Phi \in \Gamma(\mathscr{E})\,,\ m \in \mathbb{R}\,.$$

Here, $\mathscr{E}$ is a real or complex vector bundle over the Lorentzian spacetime manifold M with metric g, and $\mathcal{T}$ is a linear differential operator while $\Gamma(\mathscr{E})$ denotes the space of smooth sections of $\mathscr{E}$. m is the physical mass and may or may not be vanishing.

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Based on previous investigations on Buchdahl's higher spin equations (cf. Mühlhoff, 2007), this work started from the idea of constructing **higher spin field equations as twisted Dirac operators.** The Dirac operator is the most prominent representative of the class of prenormally hyperbolic first order differential operators, which were shown to have unique advanced and retarded Green's functions and a well-posed Cauchy problem on globally hyperbolic Lorentzian spacetime manifolds in Mühlhoff (2011).

In this paper we shall show: The twist of a prenormally hyperbolic operator remains prenormally hyperbolic. And if there is a Hermitian metric on the twisted bundle, it also allows a natural **construction of Hermitian product** on the vector space of compactly supported Cauchy initial data, which is **independent of the choice of Cauchy hypersurface** on which the initial data are located. So far we are generalizing the well-known spin $\frac{1}{2}$ construction by Dimock (1982). However, unlike in the case of spin $\frac{1}{2}$, the Hermitian product on the space of compactly supported Cauchy data will in general not be positive-definite, so that in general there is no resulting pre-Hilbert space structure and a CAR-algebraic quantization construction will not be immediate (cf. Mühlhoff, 2011, sec. V for a brief outline of the quantization construction using CAR-algebras. Cf. also the recent preprint Bär and Ginoux, 2011 for related results and algebraic quantization in a more general setting.)

Moreover, in this paper we will single out one particular twisted Dirac operator for **Fermionic fields,** which allows a canonical Hermitian bundle product which comes from a **generalized Dirac adjoint** on the higher spin bundle. However, these operators will be of indefinite type with respect to this bundle product, so that a CAR-algebraic quantization construction in the style of Dimock (1982) turns out to fail.

The search for a differential operator on higher spin fields has a long history. Dirac (1936) himself presented such attempts in flat spacetime but which were found inconsistent (i. e. integrable only under unacceptably strong assumptions) in the setting of minimal coupling to an electromagnetic field if spin > 1 by Fierz and Pauli (1939). Minimal coupling to gravity (by naively replacing partial derivatives by covariant derivatives on a curved spacetime background) was studied by Buchdahl (1962). By adding to the operator a correction term of order 0 in Buchdahl (1982a, 1982b), he was able to make the minimally coupled *massive* equations integrable for arbitrary spin under certain *constraint conditions* (for a review and additional considerations in modern language, cf. Mühlhoff, 2007). Wünsch (1985) and Illge (1993, 1996), Illge and Schimming (1999) also worked on Buchdahl's equations, but – though they are consistent (with constraints) – a natural construction of Hermitian scalar product on the respective spinor bundle which yields a pre-Hilbert space structure on the space of solutions to the Cauchy problem with compactly supported initial data, could not be established.

Dealing with spinor fields on manifolds requires a sophisticated piece of machinery in terms of notation and geometric formalism. To make this document accessible to both theoretical physicists accustomed to index notation as well as to mathematicians with a background in spin geometry, as a general principle every central result shall be stated both in an index- as well as in a mathematical (non-index) notation. In proofs and calculations though we shall often completely rely on index formalism where this brings significant notational benefits.

But anyway, the index formalism adopted in this document is an **abstract index notation.** All expressions (with or without indices) are invariant if not otherwise stated (cf. also the remarks in appendix 5). To make this document more readable to a wider audience, there is a compact but solid introduction to spinor formalism in the appendix.

II. NOTATION AND CONVENTIONS

A. Conventions

Geometric background: Throughout the document, let (M, g) be a time-oriented and space-oriented, 4-dimensional, globally hyperbolic, Lorentzian manifold of signature $(+\,-\,-\,-)$. In particular, this implies that (M, g) is oriented and connected, satisfies the

strong causality condition and has a spin structure (more details are summarized in Mühlhoff (2007); for the concept of global hyperbolicity cf. Bär *et al.* (2007), existence of a spin structure was shown in Geroch, 1970). We shall refer to (M, g) as **spacetime manifold.**

Vector bundles: For a vector bundle $\mathscr{E}$ on M, $\Gamma(\mathscr{E})$ is the space of C^∞- (i. e. smooth) sections, $\Gamma_0(\mathscr{E})$ the space of compactly supported C^∞-sections. $\mathscr{E}^*$ denotes the dual bundle, $\mathrm{End}(\mathscr{E})$ the bundle of endomorphisms and $\mathrm{GL}(\mathscr{E})$ the bundle of automorphisms of $\mathscr{E}$. By $\mathrm{Id}_{\mathscr{E}} \in \Gamma(\mathrm{GL}(\mathscr{E}))$ we denote the identity map on the fibers. For a point $m \in M$, the fiber of $\mathscr{E}$ over m is denoted by $\mathscr{E}_m$. A covariant derivatives on $\mathscr{E}$ will be denoted by $\nabla^{\mathscr{E}}$ if it is important to distinguish it from other covariant derivatives. The **tangent bundle** on (M, g) will be denoted TM, the co-tangent bundle T^*M. We always assume TM and T^*M to be equipped with the metric induced **Levi-Civita covariant derivative,** denoted by ∇.

Indices: Indices for vectors, tensors and spinors are denoted as superscripts, indices for co-vectors, co-tensors and co-spinors as subscripts. Latin lower case letters $a, b, c, \ldots$ from the beginning of the alphabet denote abstract **(co-)vector/tensor indices.** The subscript indices i, k, l are used as numerical counting indices running from 1 (i. e., they take values $1, 2, 3, \ldots$), e. g. the i-th Pauli spin matrix will be denoted $\tilde{\sigma}_i$, for $i = 1, 2, 3$. To distinguish the counting index i from the imaginary number $\boldsymbol{i} \in \mathbb{C}$, the latter is printed in bold face. Latin capital letters $A, B, C, \ldots$, denote abstract indices of **positive Weyl (co-)spinors,** dotted Latin capital letters $\dot{X}, \dot{Y}, \ldots$ denote abstract indices of **negative Weyl (co-)spinors** (for an introduction to 2-spinor index notation, cf. appendix 5). Finally, Greek lower case letters $\kappa, \lambda, \mu, \nu, \ldots$ are classical **spacetime component indices,** usually taking values from 0 to 3, i. e., $\mu = 0, \ldots, 3$. **Implicit contraction** (in the case of abstract indices) and **implicit summation** (in the case of classical spacetime component indices) is taken on *diagonal* pairs of similar indices, e. g. for $X = X^a \in \Gamma(TM)$, $\varphi = \varphi_a \in \Gamma(T^*M)$, we write $\varphi(X) = \varphi_a X^a$. **Index shifting** of vector/tensor indices is with respect to the metric g_{ab} on TM and the induced **inverse metric** g^{ab} on T^*M. Index shifting of spinor indices is with respect to the ε-spinors ε_{AB}, ε^{AB}, $\varepsilon_{\dot{X}\dot{Y}}$, $\varepsilon^{\dot{X}\dot{Y}}$, as described in appendix 5.

Lorentz geometry: For $m \in M$, a vector $x \in T_m M$ will be called **timelike**, if $g(x, x) > 0$ and **causal** if $g(x, x) \geq 0$. Let the time-orientation of (M, g) (which was assumed to be chosen above) be represented by a global timelike vector field $\tau \in \Gamma(TM)$. Then we call x **future pointing** or future directed, if $g(x, \tau) > 0$, **past directed** if $g(x, \tau) < 0$. A smooth curve $c\colon (-\varepsilon, \varepsilon) \to M$ in M is called future directed, past directed, timelike, causal, respectively, if its tangent vector $\dot{c}(t)$ has the respective property for all $t \in (-\varepsilon, \varepsilon)$. Finally, for $A \subseteq M$, $J_+(A)$ resp. $J_-(A)$ denotes the **causal future resp. past** of the set A, i. e. the set of points in M, which are either points in A or which can be reached from a point in A by a future resp. past directed, causal, piecewise C^1 curve. Moreover, we set $J(A) := J_+(A) \cup J_-(A)$.

B. Spinors on curved spacetime

In this section we declare the necessary geometric structures, in particular different types of spinor bundles, on our spacetime manifold (M, g). We assume the reader to be familiar with $\mathrm{SL}(2, \mathbb{C})$ spinor formalism on Minkowski vector space (i. e. "on the fiber"), an introduction to which may be found in the appendix.

1. Let $\mathscr{F}_c(M, g)$ denote the bundle of oriented and time-oriented pseudo-orthogonal frames on (M, g) **(connected frame bundle);** it is an $SO^+(1, 3) = \mathcal{L}_+^\uparrow$-principal bundle. Let a **spin structure** $\Lambda\colon \mathscr{S}(M, g) \to \mathscr{F}_c(M, g)$ be chosen, i. e. a $\mathrm{Spin}_0(1, 3) \cong \mathrm{SL}(2, \mathbb{C})$-principal bundle $\mathscr{S}(M, g)$ together with a bundle homomorphism Λ which is in each fiber the 2-1 universal covering map $\lambda\colon \mathrm{SL}(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow$ (cf. appendix 6).

2. Let $D\colon \mathrm{SL}(2, \mathbb{C}) \to \mathrm{GL}(\Delta)$ be a finite dimensional complex representation of $\mathrm{SL}(2, \mathbb{C})$ (i. e., a spinor representation). Then the **bundle of D-spinors on** M is the associated vector bundle

$$\mathscr{D} := \mathscr{S}(M, g) \times_D \Delta\,,$$

i. e. the vector bundle whose fiber at $m \in M$ consists of the orbits

$$[F_m, \psi]_D := \{(F_m \cdot S^{-1}, D(S)\,\psi) \,|\, S \in \mathrm{SL}(2,\mathbb{C})\}\,, \tag{O}$$

for $F_m \in \mathscr{S}(M,g)_m$ and $\psi \in \Delta$. Here, $\cdot S^{-1}$ denotes the right action of S^{-1} on the fiber $\mathscr{S}(M,g)_m$. Thus, for the positive resp. negative Weyl spinor representations $D^{(\frac{1}{2},0)}$ resp. $D^{(0,\frac{1}{2})}$ on $\Delta_{\frac{1}{2},0}$ resp. $\Delta_{0,\frac{1}{2}}$ (cf. appendix 4), we declare the **bundles of positive/negative Weyl spinors on** (M,g)**,**

$$\mathscr{D}^{(\frac{1}{2},0)} := \mathscr{S}(M,g) \times_{D^{(\frac{1}{2},0)}} \Delta_{\frac{1}{2},0}\,, \qquad \mathscr{D}^{(0,\frac{1}{2})} := \mathscr{S}(M,g) \times_{D^{(0,\frac{1}{2})}} \Delta_{0,\frac{1}{2}}\,.$$

Then the **bundle of Dirac spinors on** (M,g) is declared as

$$\mathscr{D}^D := \mathscr{D}^{\frac{1}{2},0} \oplus (\mathscr{D}^{0,\frac{1}{2}})^*\,.$$

Moreover, we shall often be using the **totally symmetric bundles**

$$\tilde{\mathscr{D}}^{(\frac{k}{2},\frac{l}{2})} := \mathscr{S}(M,g) \times_{\tilde{D}^{(\frac{k}{2},\frac{l}{2}}} \tilde{\Delta}_{\frac{k}{2},\frac{l}{2}}\,, \qquad k,l \in \mathbb{N}\,,$$

for the representations

$$\tilde{D}^{(\frac{k}{2},\frac{l}{2})} := D^{(\frac{k}{2},0)} \otimes \big(D^{(0,\frac{l}{2})}\big)^* = \big(D^{(\frac{1}{2},0)}\big)^{\vee k} \otimes \big(\big(D^{(0,\frac{1}{2})}\big)^*\big)^{\vee l}\,,$$

where ${}^{\vee k}$ denotes the k-fold symmetrized tensor product. Notice, $D^{(\frac{k}{2},\frac{l}{2})} \cong \tilde{D}^{(\frac{k}{2},\frac{l}{2})}$, but these representations do not equal (the first has the dotted indices as spinor, the other as co-spinor indices; cf. appendix 4).

Notice, it is easily seen that forming direct sums, tensor products, complex conjugates and duals of the spinor bundles is compatible with first performing the analog operations on the level of representations and then building the associated vector bundles. For example:

$$\mathscr{D}^{\frac{k}{2},0} \vee \mathscr{D}^{\frac{k'}{2},0} = \mathscr{D}^{\frac{k+k'}{2},0}\,, \qquad (\mathscr{D}^D)^* = \mathscr{S}(M,g) \times_{(D^D)^*} \big(\Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}}\big)^*\,,$$

where $\vee$ denotes the symmetrized tensor product.

3. In analogy to the fiber objects ε^{AB}, ${\sigma_a}^{A\dot{X}}$ and γ_a (cf. appendices 5, 6, 7), we shall now declare objects

$$\hat{\varepsilon}^{AB} \in \Gamma(\mathscr{D}^{\frac{1}{2},0} \otimes \mathscr{D}^{\frac{1}{2},0})\,, \quad {\hat{\sigma}_a}{}^{A\dot{X}} \in \Gamma(T^*M \otimes (\mathscr{D}^{\frac{1}{2},\frac{1}{2}})^*)\,, \quad \hat{\gamma}_a \in \Gamma(T^*M \otimes \mathrm{GL}(\mathscr{D}^D))$$

on (M,g). To this end, let first F be a local section of $\mathscr{S}(M,g)$ on an open neighborhood $U \subseteq M$. By Λ, F gets mapped to a local section of $\mathscr{F}_c(M,g)$, which constitutes a local pseudo-orthogonal tetrad field $(\hat{b}_0, \hat{b}_1, \hat{b}_2, \hat{b}_3) := \Lambda \circ F$. Moreover, let (E_1, E_2) denote the standard basis of $\Delta_{\frac{1}{2},0} = \mathbb{C}^2$. Then F induces local sections $\hat{E}_1, \hat{E}_2$ of $\mathscr{D}^{\frac{1}{2},0}$ by setting

$$\hat{E}_i := [F, E_i]_{D^{(\frac{1}{2},0)}}\,, \quad i = 1,2\,.$$

Now, let the totally anti-symmetric spinor $\varepsilon^{AB} \in \Delta_{\frac{1}{2},0} \otimes \Delta_{\frac{1}{2},0}$ from appendix 5 be given in components $\varepsilon^{\kappa\lambda}$ with respect to the (E_1, E_2). Then we define the spinor field $\hat{\varepsilon}^{AB}$ locally by

$$\hat{\varepsilon}^{AB} = \varepsilon^{\kappa\lambda}\, \hat{E}_\kappa \otimes \hat{E}_\lambda\,, \qquad \kappa,\lambda = 1,2\,.$$

(In the same way, we declare ε_{AB}; then we obtain $\varepsilon^{\dot{X}\dot{Y}}$ and $\varepsilon_{\dot{X}\dot{Y}}$ by complex conjugation.) Analogously, let the σ-spinor-tensor ${\sigma_a}^{A\dot{X}}$ from appendix 6 be given in component representation ${\sigma_\mu}^{\kappa\dot{\lambda}}$ with respect to the standard bases $(e^0, \ldots, e^3)$ of $(\mathbb{R}^4, \eta)^*_{\mathbb{C}}$,

(E_1, E_2) of $\Delta_{\frac{1}{2},0}$ and $(\bar{E}_1, \bar{E}_2)$ of $\Delta_{0,\frac{1}{2}} = \overline{\Delta_{\frac{1}{2},0}}$. Then we define the σ-tensor spinor field $\hat{\sigma}_a{}^{A\dot{X}}$ locally by

$$\hat{\sigma}_a{}^{A\dot{X}} = \sigma_\mu{}^{\kappa\dot{\lambda}}\,\hat{b}^\mu \otimes \hat{E}_\kappa \otimes \hat{\bar{E}}_{\dot{\lambda}}\,.$$

γ_a can be constructed in an analogous fashion by using $(E_1, E_2, \bar{E}^1, \bar{E}^2)$ as basis of Δ_D. Notice that in appendix 1, we chose the Weyl representation as spinor representation of $\mathrm{Cl}^c_{1,3}$. Though our theory does not depend on this choice (because all spinor representations of $\mathrm{Cl}^c_{1,3}$ are equivalent), it has the benefit that the γ-tensor spinor (cf. appendix 7) is given by

$$\hat{\gamma}_a = \begin{pmatrix} 0 & \hat{\sigma}_a{}^{A\dot{X}} \\ \hat{\sigma}_{a\,\dot{X}A} & 0 \end{pmatrix} \in \Gamma\left(T^*M \otimes \mathrm{GL}\left(\mathscr{D}^{\frac{1}{2},0} \oplus (\mathscr{D}^{0,\frac{1}{2}})^*\right)\right)\,,$$

and thus enables easy transition between Dirac- and 2-spinor formalism.

Finally, it follows from the invariance of ε, σ and γ under simultaneous reference frame transformations (cf. appendix 8) together with equation (O) above that these constructions do not depend on the choice of local section F. Thus, all these objects are well-defined *globally.* Moreover we shall henceforth drop the hat on the bundle objects and just write ε^{AB}, $\sigma_a{}^{A\dot{X}}$ and γ_a.

4. It is well known that the spin structure $\mathscr{S}(M,g)$ bears a connection naturally induced by the Levi-Civita connection on the connected frame bundle $\mathscr{F}_c(M,g)$. This connection induces covariant derivatives on all associated vector bundles (particularly, on all spinor bundles), which are again referred to as **Levi-Civita covariant derivatives,** and which are all **compatible** in the sense that they respect forming tensor products, direct sums, dual bundles or complex conjugates of the bundles and on the level of representations. Therefore we may denote all Levi-Civita covariant derivatives on all the spinor bundles by ∇_a without ambiguity. (In detail, this is elaborated in Mühlhoff, 2007.)

   Moreover, it can be shown that the fields $\sigma_a{}^{A\dot{X}}$, γ_a, ε^{AB}, ε_{AB}, $\varepsilon^{\dot{X}\dot{Y}}$ and $\varepsilon_{\dot{X}\dot{Y}}$ on M are all **parallel** with respect to the respective covariant derivatives, i. e.

$$\nabla_a\,\sigma_b{}^{A\dot{X}} \equiv 0\,, \qquad \nabla_a\,\gamma_b \equiv 0\,, \qquad \nabla_a\,\varepsilon^{AB} \equiv 0\,.$$

   This is again a consequence of the invariance of σ, ε and γ under synchronous reference frame transformations, cf. appendix 8. Notice that this result admits **implicit index shifting** of spinor indices by means of ε, in the way described in appendix 5.

5. We remind that **index shifting** of spinor indices is done with respect to the antisymmetric ε, as described in appendix 5. Moreover, for vector fields $X = X^a \in \Gamma(TM)$ and co-vector fields $\varphi_a \in \Gamma(T^*M)$, we use the notation

$$X^{A\dot{X}} := X^a\,\sigma_a{}^{A\dot{X}}\,, \qquad \varphi_{A\dot{X}} := \alpha_a\,\sigma^a{}_{A\dot{X}}\,.$$

   (Sometimes this is referred to as Dirac slash notation, though we will omit the slash.) In particular, for ∇_a we may write

$$\nabla_{A\dot{X}} = \sigma^a{}_{A\dot{X}}\nabla_a\,.$$

C. Differential operators and the Dirac equation on curved spacetimes

1. Let $\mathscr{E}$ be a general complex vector bundle on (M,g) with covariant derivative $\nabla^{\mathscr{E}}$. For linear differential operators $\mathcal{P}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ on sections of $\mathscr{E}$ we generally adopt the terminology used in Bär *et al.* (2007). Recall that for $\mathcal{P}$ of order k, the **principal symbol** $s_{\mathcal{P}}$ is an object of the type

$$s_P \in \Gamma\left((T^*M)^{\vee k} \otimes \mathrm{End}(\mathcal{E})\right)\,,$$

where $\vee k$ denotes the symmetrized k-fold tensor product. If $\mathcal{P}$ is of first order, the **principal part** (i. e. the highest order part) of $\mathcal{P}$ with respect to the covariant derivative $\nabla^{\mathscr{E}}$ may be written as

$$\Phi \mapsto (s_{\mathcal{P}})^a \nabla_a^{\mathcal{E}} \Phi\,, \qquad \Phi \in \Gamma(\mathscr{E})\,.$$

(Notice, the principal symbol is independent of the covariant derivative $\nabla^{\mathscr{E}}$, while the principal part is not.)

2. A second order linear differential operator $\mathcal{L}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ is called **normally hyperbolic,** if its principal symbol σ_L "is given by the metric":

$$\forall x \in M\, \forall \xi \in T_x^* M\colon \sigma_L(\xi,\xi) = g(\xi,\xi)\ \mathrm{Id}_{\mathscr{E}_x}\,,$$

Equivalently this means that L can locally be written as

$$L\Phi = g^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu} \Phi + \text{lower order derivatives of } \Phi\,, \quad \Phi \in \Gamma(\mathscr{E})\,.$$

A *first* order linear differential operator $\mathcal{P}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ is called **prenormally hyperbolic,** if there is another first order linear differential operator $\mathcal{Q}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$, such that $\mathcal{P}\mathcal{Q}$ is normally hyperbolic. This concept was introduced in Mühlhoff (2011) and is the central premise for the Cauchy problem of a first order operator $\mathcal{P}$ (on a globally hyperbolic manifold) to be solvable (i. e., well-posed).

3. The **Dirac operator** on the Dirac spinor bundle $\mathscr{D}^D$ is the first order linear differential operator $\mathcal{D}$ given by

$$\mathcal{D}\Psi = \gamma^a \nabla_a \Psi\,, \qquad \Psi \in \Gamma(\mathscr{D}^D)\,.$$

The associated spin $\frac{1}{2}$ **Dirac field equation** for Dirac fields of mass $m \in \mathbb{R}$ reads:

$$0 = \mathcal{D}\Phi + \boldsymbol{i}\, m\, \Phi\,, \qquad \Psi \in \Gamma(\mathscr{D}^D)\,.$$

Using the chiral decomposition $\mathscr{D}^D = \mathscr{D}^{\frac{1}{2},0} \oplus (\mathscr{D}^{0,\frac{1}{2}})^*$, and writing $\Psi = \big((\psi_1)^A, (\psi_2)_{\dot{X}}\big)^{\mathrm{tr}}$ for $\psi_1 \in \Gamma\big(\mathscr{D}^{\frac{1}{2},0}\big)$ and $\psi_2 \in \Gamma\big((\mathscr{D}^{0,\frac{1}{2}})^*\big)$, we find

$$\begin{aligned} 0 = \mathcal{D}\Psi + \boldsymbol{i}\, m\, \Psi &= \begin{pmatrix} 0 & {\sigma_a}^{A\dot{X}} \\ \sigma_{a\,\dot{X}A} & 0 \end{pmatrix} \begin{pmatrix} \nabla_a\, (\psi_1)^A \\ \nabla_a\, (\psi_2)_{\dot{X}} \end{pmatrix} + \boldsymbol{i}\, m \begin{pmatrix} (\psi_1)^A \\ (\psi_2)_{\dot{X}} \end{pmatrix} \\ \Leftrightarrow \quad & \begin{cases} 0 = \nabla^{A\dot{X}}\, (\psi_2)_{\dot{X}} + \boldsymbol{i}\, m\, (\psi_1)^A \\ 0 = \nabla_{\dot{X}A}\, (\psi_1)^A + \boldsymbol{i}\, m\, (\psi_2)_{\dot{X}} \end{cases} \end{aligned}.$$

This shows how the Dirac equation decomposes into a system of the two coupled Weyl equations.

III. FERMIONIC QUANTUM FIELDS OF HIGHER SPIN

It is our goal to investigate field equations for spinor fields of arbitrarily high spin on our Lorentzian spacetime manifold (M, g). Loosely speaking, this could mean for fields of the general form $\Psi^{A_1 \dots A_k \dot{X}_1 \dots \dot{X}_l}$, i. e. "with a higher number of dotted and undotted indices", whence $\frac{k+l}{2}$ will then be the spin value. Such a field is called **Fermionic,** if the spin $\frac{k+l}{2}$ is half-integral. We shall specialize to Fermionic fields later, but not from the beginning.

Considering higher spin fields has a long history and there have been various approaches, as pointed out in section I. In the works of **Buchdahl** (1982a, 1982b) and **Wünsch** (1985), fully symmetric generalized Dirac spinors

$$\Psi = \begin{pmatrix} (\psi_1)^{AA_1 \dots A_k}{}_{\dot{X}_1 \dots \dot{X}_l} \\ (\psi_2)_{\dot{X}}{}^{A_1 \dots A_k}{}_{\dot{X}_1 \dots \dot{X}_l} \end{pmatrix} = \begin{pmatrix} (\psi_1)^{(AA_1 \dots A_k)}{}_{(\dot{X}_1 \dots \dot{X}_l)} \\ (\psi_2)_{(\dot{X}}{}^{(A_1 \dots A_k)}{}_{\dot{X}_1 \dots \dot{X}_l)} \end{pmatrix}$$

were considered, but in case of Wünsch, the principal symbol of his first order operator is not invertible in normal direction (and thus the operator fails to be prenormally hyperbolic), in case of Buchdahl the enforcement of symmetry in all indices leads to additional and seemingly unnatural constraint conditions (which is not a fatal problem though). Moreover, in both cases, there doesn't seem to exist a canonical Hermitian scalar product on the bundle with respect to which the differential operators are not of positive definite type, and thus a canonical quantization of the system using CAR-algebras appears not to be feasible. (Buchdahl's and Wünsch's operators were studied in detail in Mühlhoff, 2007).

The **basic idea** of the present approach is to see in higher spin fields the structure of **twisted Dirac fields,** i. e. to consider bundles of the type

$$\mathscr{M}^{\frac{k}{2},\frac{l}{2}} := \mathscr{D}^D \otimes \tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}} = \left(\mathscr{D}^{\frac{1}{2},0} \oplus (\mathscr{D}^{0,\frac{1}{2}})^*\right) \otimes \tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}, \qquad k, l \in \mathbb{N}\,.$$

Notice that the fibers of the bundle $\mathscr{M}^{\frac{k}{2},\frac{l}{2}}$ are isomorphic to

$$\Delta_D \otimes \tilde{\Delta}_{\frac{k}{2},\frac{l}{2}} = (\Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}}) \otimes \tilde{\Delta}_{\frac{k}{2},\frac{l}{2}}$$

and that the spin value of this representation is $\frac{k+l+1}{2}$. Sections $\Psi \in \Gamma(\mathscr{M}^{\frac{k}{2},\frac{l}{2}})$ are of the form

$$\Psi = \begin{pmatrix} (\psi_1)^{AA_1 \dots A_k}{}_{\dot{X}_1 \dots \dot{X}_l} \\ (\psi_2)_{\dot{X}}{}^{A_1 \dots A_k}{}_{\dot{X}_1 \dots \dot{X}_l} \end{pmatrix}.$$

A. Twisted Prenormally Hyperbolic Operators

In this section we find as an auxiliary proposition that twists of prenormally hyperbolic first order differential operators are again prenormally hyperbolic. This general result will be applied to the higher spin operators introduced in the subsequent section. The concept of prenormal hyperbolicity was introduced in Mühlhoff (2011) as the central condition for the existence of unique advanced and retarded Green's operators and of solutions to the Cauchy problem for first order operators.

Let $\mathscr{E}$ and $\mathscr{F}$ be complex vector bundles on our spacetime manifold (M, g), equipped with covariant derivatives $\nabla^{\mathscr{E}}$ and $\nabla^{\mathscr{F}}$, respectively. Let $\mathcal{P}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ be a linear differential operator on $\mathscr{E}$. Then the **twisted operator** $\mathcal{P}^{\mathscr{F}}\colon \Gamma(\mathscr{E} \otimes \mathscr{F}) \to \Gamma(\mathscr{E} \otimes \mathscr{F})$ is the linear differential operator which is for simple tensors $\psi \otimes \alpha \in \Gamma(\mathscr{E} \otimes \mathscr{F})$, $\psi \in \Gamma(\mathscr{E})$, $\alpha \in \Gamma(\mathscr{F})$, given by

$$\mathcal{P}^{\mathscr{F}}(\psi \otimes \alpha) := \mathcal{P}\psi \otimes \alpha + (s_{\mathcal{P}})^a \psi \otimes \nabla^{\mathscr{F}}_a \alpha\,,$$

where $s_{\mathcal{P}} \in \Gamma(T^*M \otimes \mathrm{End}(\mathscr{E}))$ denotes the principal symbol of $\mathcal{P}$. Notice that the principal part (i. e. the highest order part) of $\mathcal{P}^{\mathscr{F}}$ is

$$\psi \otimes \alpha \;\mapsto\; (s_{\mathcal{P}})^a \nabla^{\mathscr{E}}_a \psi \otimes \alpha + (s_{\mathcal{P}})^a \psi \otimes \nabla^{\mathscr{F}}_a \alpha = \left((s_{\mathcal{P}})^a \otimes \mathrm{Id}_{\mathscr{F}}\right) \nabla^{\mathscr{E}\otimes\mathscr{F}}_a (\psi \otimes \alpha)\,, \qquad (*)$$

where $\nabla^{\mathscr{E}\otimes\mathscr{F}}$ denotes the induced tensor product covariant derivative on $\mathscr{E}\otimes\mathscr{F}$, given by

$$\nabla_a^{\mathscr{E}\otimes\mathscr{F}}(\psi\otimes\alpha) = \nabla_a^{\mathscr{E}}\psi\otimes\alpha + \psi\otimes\nabla_a^{\mathscr{F}}\alpha\,.$$

From $(*)$ we see that the principal symbol of the twisted operator $\mathcal{P}^{\mathscr{F}}$ is given by

$$s_{\mathcal{P}^{\mathscr{F}}} = s_{\mathcal{P}}\otimes \mathrm{Id}_{\mathscr{F}} \;\in\; \Gamma(T^*M\otimes \mathrm{End}(\mathscr{E}\otimes\mathscr{F}))\,.$$

Proposition 1. Let $\mathscr{E}$ and $\mathscr{F}$ be complex vector bundles on (M,g). Let $\mathcal{P}\colon\Gamma(\mathscr{E})\to\Gamma(\mathscr{E})$ be a prenormally hyperbolic differential operator of first order. Then the twisted operator $\mathcal{P}^{\mathscr{F}}$ on sections of $\mathscr{E}\otimes\mathscr{F}$ is again prenormally hyperbolic.

Proof. Let $\mathcal{Q}\colon\Gamma(\mathscr{E})\to\Gamma(\mathscr{E})$ be a second first order operator such that $\mathcal{P},\mathcal{Q}$ form a complementary pair (terminology cf. Mühlhoff, 2011, definition 1). Then $\mathcal{PQ}$ is of second order and normally hyperbolic, i. e. for all $\xi\in T^*M$,

$$s_{\mathcal{PQ}}(\xi) = s_{\mathcal{P}}(\xi)\, s_q(\xi) = g(\xi,\xi)\ \mathrm{Id}_{\mathscr{E}}\ .$$

Notice that the twist $\mathcal{Q}^{\mathscr{F}}$ has principal symbol $s_{\mathcal{Q}^{\mathscr{F}}}(\xi) = s_{\mathcal{Q}}(\xi)\otimes\mathrm{Id}_{\mathscr{F}}$ and thus we find

$$\begin{aligned} s_{\mathcal{P}^{\mathscr{F}}\mathcal{Q}^{\mathscr{F}}}(\xi) &= (s_{\mathcal{P}}(\xi)\otimes\mathrm{Id}_{\mathscr{F}})\,(s_{\mathcal{Q}}(\xi)\otimes\mathrm{Id}_{\mathscr{F}})\\ &= (s_{\mathcal{P}}(\xi)\,s_{\mathcal{Q}}(\xi))\otimes\mathrm{Id}_{\mathscr{E}\otimes\mathscr{F}} = g(\xi,\xi)\ \mathrm{Id}_{\mathscr{E}\otimes\mathscr{F}}\ . \end{aligned}$$

This means that $\mathcal{P}^{\mathscr{F}}\mathcal{Q}^{\mathscr{F}}$ is normally hyperbolic and thus, by definition of prenormal hyperbolicity, $\mathcal{P}^{\mathscr{F}}$ is prenormally hyperbolic. ☐

B. Higher spin quantum fields as twisted Dirac fields

We now apply the general considerations above to the situation where $\mathscr{E}$ is the Dirac spinor bundle, $\mathscr{E} = \mathscr{D}^D = \mathscr{D}^{\frac12,0}\oplus(\mathscr{D}^{0,\frac12})^*$, equipped with the Dirac operator

$$\mathcal{D}(\psi) = \gamma^a\nabla_a\psi\,,\qquad \psi\in\Gamma(\mathscr{D}^D)$$

(which is easily seen to be prenormally hyperbolic, cf. Mühlhoff, 2011). For $\mathscr{F}$ we may take any spinor bundle, i. e. arbitrary tensor products and direct sums of $\mathscr{D}^{\frac12,0}$ and $(\mathscr{D}^{0,\frac12})^*$. But as all of them can be broken down into direct sums of symmetric spinors, we shall restrict our attention here to $\mathscr{F} = \tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}$, for yet arbitrary $k,l\in\mathbb{N}$.

Definition 1 (higher spin bundles and operators). For $k,l\in\mathbb{N}$, define the higher spinor bundles

$$\mathscr{M}^{k,l} := \mathscr{D}^D\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}} = \big(\mathscr{D}^{\frac12,0}\oplus(\mathscr{D}^{0,\frac12})^*\big)\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}$$

and the higher spin first order differential operators

$$\mathcal{T}^{k,l} := \mathcal{D}^{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}}\,,\qquad \mathcal{T}^{k,l}\colon\Gamma(\mathscr{M}^{k,l})\to\Gamma(\mathscr{M}^{k,l})\,.$$

(In words: $\mathcal{T}^{k,l}$ is the spin $\frac12$ Dirac operator twisted by the spin $\frac{k+l}{2}$ bundle $\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}$.) Then for every $k,l\in\mathbb{N}$ and for every physical mass $m\in\mathbb{R}$ we have the spin $s=\frac{k+l+1}{2}$ field equation

$$\mathcal{T}^{k,l}\Phi + \boldsymbol{i}\,m\,\Phi = 0\,,\qquad \Phi\in\Gamma(\mathscr{M}^{k,l})\,.$$

Remark 1 (principal symbol of $\mathcal{T}^{k,l}$). The principal symbol of the Dirac operator $\mathcal{D}$ is $(s_{\mathcal{D}})^a = \gamma^a$. Thus, the principal symbol of $\mathcal{T}^{k,l}$ reads

$$(s_{\mathcal{T}^{k,l}})^a = \gamma^a\otimes\mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}}$$

and we can write

$$\mathcal{T}^{k,l}\Phi = (\gamma^a \otimes \mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}})\,\nabla_a\Phi \overset{(2)}{=} \gamma^a\nabla_a\psi\otimes\alpha + \gamma^a\psi\otimes\nabla_a\alpha\,,$$

while the second equality (2) holds only for simple tensors $\Phi = \psi\otimes\alpha \in \Gamma(\mathscr{M}^{k,l})$, $\psi \in \Gamma(\mathscr{D}^D)$, $\alpha \in \Gamma(\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}})$. Here we denoted all covariant derivatives by ∇, which is possible without ambiguity as the covariant derivatives on all 2-spinor bundles are compatible, cf. section II B.

Remark 2 (chiral decomposition of $\mathcal{T}^{k,l}$). The Dirac operator $\mathcal{D}$ has the chiral decomposition into positive and negative Weyl operators:

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_- \\ \mathcal{D}_+ & 0 \end{pmatrix}, \qquad \begin{aligned} \mathcal{D}_+ &: \Gamma(\mathscr{D}^{\frac{1}{2},0}) \to \Gamma\big((\mathscr{D}^{0,\frac{1}{2}})^*\big)\,, \\ \mathcal{D}_- &: \Gamma\big((\mathscr{D}^{0,\frac{1}{2}})^*\big) \to \Gamma(\mathscr{D}^{\frac{1}{2},0})\,. \end{aligned}$$

This decomposition is inherited by the twisted Dirac operators $\mathcal{T}^{k,l}$:

$$\mathcal{T}^{k,l} = \begin{pmatrix} 0 & \mathcal{T}^{k,l}_- \\ \mathcal{T}^{k,l}_+ & 0 \end{pmatrix}, \qquad \begin{aligned} \mathcal{T}^{k,l}_+ &= (\mathcal{D}_+)^{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}}\,, \\ \mathcal{T}^{k,l}_- &= (\mathcal{D}_-)^{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}}\,. \end{aligned}$$

As in the case of the Dirac operator $\mathcal{D}$, the chiral decomposition of $\mathcal{T}^{k,l}$ is reflected in 2-spinor notation of $\mathcal{T}^{k,l}$:

Remark 3 (higher spin fields in 2-spinor notation). Notice that

$$\mathscr{M}^{k,l} \cong \mathscr{D}^{\frac{1}{2},0}\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}} \;\oplus\; (\mathscr{D}^{0,\frac{1}{2}})^*\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}\,.$$

Thus, in 2-spinor notation, a section $\Phi \in \Gamma(\mathscr{M}^{\frac{k}{2},\frac{l}{2}})$ is written as

$$\Phi = \begin{pmatrix} (\varphi_1)^{AA_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \\ (\varphi_2)_{\dot{X}}{}^{A_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \end{pmatrix}, \qquad \varphi_1 \in \Gamma(\mathscr{D}^{\frac{1}{2},0}\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}),\ \varphi_2 \in \Gamma\big((\mathscr{D}^{0,\frac{1}{2}})^*\otimes\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}\big)\,,$$

and $\mathcal{T}^{k,l}\Phi$ is given by

$$\mathcal{T}^{k,l}\Phi = \underbrace{\left[\begin{pmatrix} 0 & \sigma^{a\,A\dot{X}} \\ \sigma^a{}_{\dot{X}A} & 0 \end{pmatrix}\otimes\mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}}\right]}_{\text{principal symbol } s_{\mathcal{T}^{k,l}}} \begin{pmatrix} \nabla_a(\varphi_1)^{AA_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \\ \nabla_a(\varphi_2)_{\dot{X}}{}^{A_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \end{pmatrix}$$

$$= \begin{pmatrix} \nabla^{A\dot{X}}(\varphi_2)_{\dot{X}}{}^{A_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \\ \nabla_{\dot{X}A}(\varphi_1)^{AA_1\dots A_k}{}_{\dot{X}_1\dots\dot{X}_l} \end{pmatrix} = \begin{pmatrix} \mathcal{T}^{k,l}_-(\varphi_2) \\ \mathcal{T}^{k,l}_+(\varphi_1) \end{pmatrix}$$

Finally, as an immediate application of the general result proposition 1, we may write down:

Theorem 1 (prenormal hyperbolicity, Green's functions and Cauchy problem). For every $k,l \in \mathbb{N}$, we find for the higher spin first order differential operator $\mathcal{T}^{k,l}$ on sections of $\mathscr{M}^{k,l}$:

(a) $\mathcal{T}^{k,l}$ is prenormally hyperbolic in the sense of Mühlhoff (2011).

(b) There are unique advanced and retarded Green's operators (fundamental solutions)

$$G^{k,l}_\pm : \Gamma_0(\mathscr{M}^{k,l}) \to \Gamma(\mathscr{M}^{k,l})$$

for $\mathcal{T}^{k,l}$ (while Γ_0 denoted the space of compactly supported sections).

(c) Let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface and $m \in \mathbb{R}$ physical mass. Then the Cauchy problem

$$(\mathcal{T}^{k,l}) \qquad \begin{cases} \mathcal{T}^{k,l}\Phi + \boldsymbol{i}\, m\, \Phi = 0, & \Phi \in \Gamma(\mathscr{M}^{k,l}) \\ \Phi|_\Sigma = \Phi_0 \end{cases}$$

has a unique solution for every initial datum $\Phi_0 \in \Gamma_0(\mathscr{M}^{k,l}|_\Sigma)$. Moreover, this solution satisfies $\operatorname{supp}\Phi \subseteq J(\operatorname{supp}\Phi_0)$.

Proof. (a) is a corollary of proposition 1. After this is established, the general theory presented in Mühlhoff (2011) can be applied (particularly, Mühlhoff, 2011, theorems 1, 2) to yield (b) and (c). ☐

C. Fermionic higher spin operators of indefinite type

Let $(\mathscr{E}, \langle\cdot,\cdot\rangle)$ be a Hermitian vector bundle on (M, g), i. e. a complex vector bundle equipped with a Hermitian scalar product and compatible covariant derivative $\nabla^{\mathscr{E}}$. A linear differential operator $\mathcal{P}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ with principal symbol $s_\mathcal{P}$ is called **of positive-definite respectively of indefinite type with respect to $\langle\cdot,\cdot\rangle$,**[1] if for every timelike, future directed co-vector field $0 \neq \xi \in \Gamma(T^*M)$, the sesqui-linear form[2]

$$(\Psi, \Phi)_\xi := \langle \Psi, \xi_a (s_\mathcal{P})^a \Phi\rangle\,, \qquad \Psi, \Phi \in \Gamma(\mathscr{E})$$

is (everywhere) positive-definite/indefinite, respectively.

Remark 4. It is easy to see that $(\cdot,\cdot)_\xi$ is Hermitian if and only if $s_\mathcal{P}$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle$ (i. e., $\forall \xi \in \Gamma(T^*M)\, \forall \Phi, \Psi \in \Gamma(\mathscr{E})\colon \langle \Phi, s_\mathcal{P}(\xi)\Psi\rangle = \langle s_\mathcal{P}(\xi)\Phi, \Psi\rangle$).

Example 1 (Dirac operator). It is well known that the spin $\frac{1}{2}$ Dirac operator $\mathcal{D}$ on $\mathscr{D}^D$ is of positive definite type with respect to the Hermitian scalar product on $\mathscr{D}^D$ given by pairing with the Dirac adjoint (cf. Dimock, 1982):

The **Dirac adjoint** is the complex anti-linear mapping ${}^+\colon \mathscr{D}^D \to (\mathscr{D}^D)^*$, given by

$$\Psi^+ := \begin{pmatrix} \bar{\varphi}_A \\ \bar{\psi}^{\dot{X}} \end{pmatrix} \in \Gamma\left((\mathscr{D}^D)^*\right) \quad \text{for} \quad \Psi = \begin{pmatrix} \psi^A \\ \varphi_{\dot{X}} \end{pmatrix} \in \Gamma(\mathscr{D}^D)\,.$$

By pairing spinors and co-spinors, this induces a Hermitian scalar product on $\mathscr{D}^D$:

$$\langle \Psi, \Phi\rangle_D := \Psi^+(\Phi)\,, \quad \Psi, \Phi \in \Gamma(\mathscr{D}^D)\,.$$

With respect to this Hermitian scalar product, $\mathcal{D}$ is of positive-definite type. This means that for every nowhere vanishing, timelike, future pointing co-vector field $\xi \in \Gamma(T^*M)$,

$$(\Psi, \Phi)_\xi := \langle \Psi, \xi_a \gamma^a \Phi\rangle_D = \Psi^+(\xi_a\gamma^a\Phi) = (\psi_2)_A \xi^{A\dot{X}} (\varphi_2)_{\dot{X}} + (\psi_1)^{\dot{X}} \xi_{\dot{X}A} (\varphi_1)^A$$

is positive definite in each fiber. (Here we used the notation $\Psi = (\psi_1, \psi_2)^{tr}$, $\Phi = (\varphi_1, \varphi_2)^{tr} \in \Gamma(\mathscr{D}^D)$ and $\xi^{A\dot{X}} = \xi^a \sigma_a{}^{A\dot{X}}$.)

Moreover, it is easily seen that γ^a, the principal symbol of the Dirac operator, is self-adjoint with respect to $\langle\cdot,\cdot\rangle_D$; hence, $(\cdot,\cdot)_\xi$ is also Hermitian.

[1] Cf. the terminology in the recent preprint Bär and Ginoux (2011).
[2] Sesqui-linear means that the form is complex anti-linear in the first and linear in the second argument.

In order to generalize the Dirac spin $\frac{1}{2}$ case, we would like to have a Hermitian product $\langle\cdot,\cdot\rangle_{\mathscr{M}^{k,l}}$ on our higher spin bundles $\mathscr{M}^{k,l}$ with respect to which the operators $\mathcal{T}^{k,l}$ are of positive-definite type. The differential operator being of positive-definite type with respect to the Hermitian product on the bundle is the central prerequisite for our construction to qualify for **quantization using a C^*-algebra representation of the canonical anti-commutation relations,** called CAR-algebra (see the well-known construction for the spin $\frac{1}{2}$ special case by Dimock, 1982 and the remarks on the general procedure in Mühlhoff, 2011, section V, as well as Araki, 1970).

We shall now demonstrate that if $k = l$, there is a natural Hermitian product on $\mathscr{M}^{k,l}$, but alas, the operators $\mathcal{T}^{k,k}$ will be *indefinite* with respect to it. Thus, we will see that a quantization of the fields presented here is *not* possible in a straight forward fashion. Notice that setting $k = l$ is the point where we restrict ourselves to **Fermionic fields,** as this implies that the spin $\frac{k+l+1}{2}$ is half-integral.

The natural Hermitian product on the twisted bundle $\mathscr{M}^{k,k} = \mathscr{D}^D \otimes \tilde{\mathscr{D}}^{\frac{k}{2},\frac{l}{2}}$ will be the tensor product of $\langle\cdot,\cdot\rangle_D$ on $\mathscr{D}^D$ as declared in example 1 and a Hermitian product $\langle\cdot,\cdot\rangle_k$ on $\tilde{\mathscr{D}}^{\frac{k}{2},\frac{k}{2}}$:

Definition 2 (generalized Dirac adjoint and Hermitian product $\langle\cdot,\cdot\rangle_{\mathscr{M}^{k,k}}$ on $\mathscr{M}^{k,k}$). Let two general spinors $\Phi, \Psi \in \Gamma(\mathscr{M}^{k,k})$ be written in 2-spinor notation as

$$\Phi = \begin{pmatrix} (\varphi_1)^{AA_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k} \\ (\varphi_2)_{\dot X}{}^{A_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} (\psi_1)^{AA_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k} \\ (\psi_2)_{\dot X}{}^{A_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k} \end{pmatrix},$$

Then, on the Fermionic higher spinor bundle $\mathscr{M}^{k,k}$ we declare the Hermitian product

$$\langle\Phi,\Psi\rangle_{\mathscr{M}^{k,k}} := \Phi^+(\Psi)\,.$$

Here, Φ^+ is a **generalized concept of Dirac adjoint** co-spinor, defined as:

$$\Phi^+ := \begin{pmatrix} (\bar\varphi_2)_A{}^{\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k} \\ (\bar\varphi_1)^{\dot X\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k} \end{pmatrix}.$$

Notice, this means that $\langle\cdot,\cdot\rangle_{\mathscr{M}^{k,k}}$ is given by

$$\begin{aligned}\langle\Phi,\Psi\rangle_{\mathscr{M}^{k,k}} = {}&(\bar\varphi_2)_A{}^{\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k}(\psi_1)^{AA_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k}\\ &+(\bar\varphi_1)^{\dot X\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k}(\psi_2)_{\dot X}{}^{A_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k}\,.\end{aligned}$$

Using the principal bundle $s_{\mathcal{T}^{k,k}}$ of $\mathcal{T}^{k,k}$, we now construct the Hermitian product

$$(\Psi,\Phi)_\xi := \langle\Psi, s_{\mathcal{T}^{k,k}}(\xi)\,\Phi\rangle_{\mathscr{M}^{k,k}}\,, \qquad \Psi,\Phi\in\Gamma(\mathscr{M}^{k,k})$$

for every nowhere vanishing, future pointing timelike co-vector field $\xi \in \Gamma(T^*M)$. In 2-spinor notation, this product reads:

$$\begin{aligned}(\Phi,\Psi)_\xi = {}&(\bar\varphi_2)_A{}^{\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k}\xi^{A\dot X}(\psi_2)_{\dot X}{}^{A_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k}\\ &+(\bar\varphi_1)^{\dot X\dot X_1\ldots\dot X_k}{}_{A_1\ldots A_k}\xi_{\dot X A}(\psi_1)^{AA_1\ldots A_k}{}_{\dot X_1\ldots\dot X_k}\,.\end{aligned}$$

Remark 5. It is easily seen that the principal symbol $s_{\mathcal{T}^{k,k}} = \gamma\otimes\mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{k}{2}}}$ of $\mathcal{T}^{k,k}$ is self-adjoint with respect to $\langle\cdot,\cdot\rangle_{\mathscr{M}^{k,k}}$. Hence, $(\cdot,\cdot)_\xi$ is Hermitian (cf. remark 4).

Remark 6 (indefiniteness of $(\cdot,\cdot)_\xi$). For higher spin, i. e. for $k \geq 2$, the Hermitian product $(\cdot,\cdot)_\xi$ is indefinite and thus, $\mathcal{T}^{k,k}$ are of indefinite type with respect to $(\cdot,\cdot)_\xi$.

Proof. It suffices to prove this in the fiber. First, let $e_0, \dots, e_3$ be the standard basis of $(\mathbb{R}^4, \eta)$, let E_1, E_2 be the standard basis of $\Delta_{\frac{1}{2},0}$, and set $\xi = e_0$. Then it is easily seen that for $\Phi = (\phi_1, \phi_2)^{\mathrm{tr}}$ with

$$(\phi_1)^{AA_1\dots A_k}{}_{\dot X_1\dots\dot X_k} = (E_1 \otimes \dots \otimes E_1)^{AA_1\dots A_k}\,(\bar E_1^* \otimes \dots \otimes \bar E_1^*)_{\dot X_1\dots\dot X_k}\,,$$
$$(\phi_2)_{\dot X}{}^{A_1\dots A_k}{}_{\dot X_1\dots\dot X_k} = (E_1 \otimes \dots \otimes E_1)^{A_1\dots A_k}\,(\bar E_1^* \otimes \dots \otimes \bar E_1^*)_{\dot X\dot X_1\dots\dot X_k}\,,$$

$(\Phi, \Phi)_\xi > 0$. On the other hand, for $\Phi = (\phi_1, \phi_2)^{\mathrm{tr}}$ with

$$\begin{aligned}(\psi_1)^{AA_1\dots A_k}{}_{\dot X_1\dots\dot X_k} &= (E_1 \otimes \dots \otimes E_1)^{AA_1\dots A_k}\,(\bar E_2^* \otimes \dots \otimes \bar E_2^*)_{\dot X_1\dots\dot X_k}\\ &\quad - (E_1)^A (E_2 \otimes \dots \otimes E_2)^{A_1\dots A_k}\,(\bar E_1^* \otimes \dots \otimes \bar E_1^*)_{\dot X_1\dots\dot X_k}\,,\\ (\psi_2)_{\dot X}{}^{A_1\dots A_k}{}_{\dot X_1\dots\dot X_k} &= (E_1 \otimes \dots \otimes E_1)^{A_1\dots A_k}\,(\bar E_1^*)_{\dot X}\,(\bar E_2^* \otimes \dots \otimes \bar E_2^*)_{\dot X\dot X_1\dots\dot X_k}\\ &\quad - (E_2 \otimes \dots \otimes E_2)^{A_1\dots A_k}\,(\bar E_1^* \otimes \dots \otimes \bar E_1^*)_{\dot X\dot X_1\dots\dot X_k}\,,\end{aligned}$$

we find $(\Psi, \Psi)_\xi < 0$. For ξ different from e_0 it is not difficult to construct analogous examples: Just transform the basis E_1, E_2 by $S \in \mathrm{SL}(2, \mathbb{C})$ when transforming ξ by $\Lambda(S)$. Then the same formulas as above will yield suitable examples. □

This finding is of course fatal to the construction of a C^*-algebra representation of the canonical anti-commutation relations (CAR-algebra). However, compared to what we achieved in previous work (cf. Mühlhoff, 2007), it is already a success that $(\cdot,\cdot)_\xi$ induces a Hermitian (indefinite) product on the space of Solutions to the field equation, $\mathcal{T}^{k,k}\Phi = 0$, which *does not depend on further choices.* This shall be demonstrated now:

For the remainder of this section, fix an arbitrary $k \in \mathbb{N}$. To simplify notation, we set $\mathcal{T} := \mathcal{T}^{k,k}$.

1. Let $\Sigma \subseteq M$ be a smooth spacelike Cauchy hypersurface. We define the **vector space of compactly supported Cauchy data** on Σ,

   $$\mathscr{H}_\Sigma := \Gamma_0(\mathscr{M}^{k,k}|_\Sigma)\,,$$

   on which we declare the product

   $$\langle\Phi_0, \Psi_0\rangle_\Sigma := \int\limits_\Sigma (\Phi_0, \Psi_0)_{\mathfrak{n}} = \int\limits_\Sigma \mathfrak{n}^a \langle\Phi, (s_{\mathcal{T}})_a\Psi\rangle_{\mathscr{M}^{k,k}}\,, \qquad \Phi_0, \Psi_0 \in \mathscr{H}_\Sigma\,,$$

   where $\mathfrak{n}$ is the future pointing unit normal vector field along Σ. $\langle\cdot,\cdot\rangle_\Sigma$ is Hermitian by remark 5 and for $k \geq 2$ (i. e., for the higher spin case) indefinite by remark 6.

2. We define the **vector space of solutions with compactly supported Cauchy data:**

   $$\mathscr{H} := \{\Phi \in \Gamma(\mathscr{M}^{k,k}) \,|\, \mathcal{T}^{k,k}\Phi = 0 \wedge \Phi|_\Sigma \in \mathscr{H}_\Sigma\}\,.$$

   This is independent of the choice of Σ, which means that for a second smooth Cauchy hypersurface $\Sigma' \subseteq M$, $\mathrm{supp}(\Phi) \cap \Sigma'$ is again compact for $\Phi \in \mathscr{H}$. This is a consequence of global hyperbolicity of (M, g), cf. Mühlhoff, 2011, corollary 1.

   There is a canonical vector space isomorphism $\Xi_\Sigma \colon \mathscr{H}_\Sigma \to \mathscr{H}$, given by assigning to $\Phi_0 \in \mathscr{H}_\Sigma$ the unique solution to the Cauchy problem $(\mathcal{T}^{k,k})$ with initial datum Φ_0. The inverse Ξ_Σ^{-1} is given by the restriction map $\Phi \mapsto \Phi|_\Sigma$.

   On $\mathscr{H}$ we obtain a Hermitian scalar product $\langle\cdot,\cdot\rangle$ by pushing $\langle\cdot,\cdot\rangle_\Sigma$ from $\mathscr{H}_\Sigma$ to $\mathscr{H}$ via Ξ_Σ:

   $$\langle\Phi, \Psi\rangle := \langle\Phi|_\Sigma, \Psi|_\Sigma\rangle_\Sigma\,, \qquad \Phi, \Psi \in \mathscr{H}\,.$$

   It is a crucial and non-trivial point that this construction is well-defined (i. e., independent of the choice of Σ). Thus, as central result of this section we shall prove:

Theorem 2. The Hermitian product $\langle\cdot,\cdot\rangle$ on $\mathscr{H}$ is independent of the choice of smooth spacelike Cauchy hypersurface $\Sigma \subseteq M$.

Notice, this result is well known for the spin $\frac{1}{2}$ special case of Dirac spinor fields, cf. Dimock (1982), but so far it was not generalized to an appropriate class of higher spin fields.

Proof. Fix arbitrary $\Phi, \Psi \in \mathscr{H}$. We have to show

$$\int_\Sigma \mathfrak{n}^a \langle \Phi, (s_{\mathcal{T}})_a \Psi \rangle_{\mathscr{M}^{k,k}} = \int_{\Sigma'} \mathfrak{n}'^a \langle \Phi, (s_{\mathcal{T}})_a \Psi \rangle_{\mathscr{M}^{k,k}} \tag{$*$}$$

for every second smooth Cauchy hypersurface $\Sigma' \subseteq M$ with future pointing unit normal vector field $\mathfrak{n}'$.

Declare the set $K := \operatorname{supp}(\Phi) \cup \operatorname{supp}(\Psi)$. It suffices to prove $(*)$ for the case that $K \cap \Sigma'$ lies completely in the future of $K \cap \Sigma$. Because if this is not the case, global hyperbolicity of (M, g) always allows us to choose a third smooth spacelike Cauchy hypersurface Σ'' such that $K \cap \Sigma''$ lies in the future of the compact set $K \cap (\Sigma \cup \Sigma')$ (use a time-parameterized foliation of (M, g) by Cauchy hypersurfaces, cf. Bernal and Sánchez, 2005). Transitivity would then yield equation $(*)$ for Σ and Σ'.

Define the vector field $X^a \in \Gamma(TM)$,

$$X^a := \langle \Phi, (s_{\mathcal{T}})^a \Psi \rangle_{\mathscr{M}^{k,k}} ,$$

and notice that because $\mathcal{T}\Phi = 0$ and $\mathcal{T}\Psi = 0$, X is divergence free:

$$\begin{aligned}
\operatorname{div} X = \nabla_a X^a &= \langle \nabla_a \Phi, (s_{\mathcal{T}})^a \Psi \rangle_{\mathscr{M}^{k,k}} + \langle \Phi, \nabla_a ((s_{\mathcal{T}})^a \Psi) \rangle_{\mathscr{M}^{k,k}} \\
&= \langle (s_{\mathcal{T}})^a \nabla_a \Phi, \Psi \rangle_{\mathscr{M}^{k,k}} + \langle \Phi, (s_{\mathcal{T}})^a \nabla_a \Psi \rangle_{\mathscr{M}^{k,k}} \\
&= \langle \mathcal{T}\Phi, \Psi \rangle_{\mathscr{M}^{k,k}} + \langle \Phi, \mathcal{T}\Psi \rangle_{\mathscr{M}^{k,k}} \\
&= 0 .
\end{aligned}$$

Here, on the first summand we used self-adjointness of $s_{\mathcal{T}}$ (remark 4), for the second summand we used that $s_{\mathcal{T}} = \gamma \otimes \mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{k}{2}}}$ is parallel (i. e., $\nabla_a(\gamma \otimes \mathrm{Id}_{\tilde{\mathscr{D}}^{\frac{k}{2},\frac{k}{2}}}) \equiv 0$; for $\nabla_a \gamma \equiv 0$ cf. section II B).

Choose Σ' such that $K \cap \Sigma'$ lies completely in the future of $K \cap \Sigma$. Then due to global hyperbolicity, the set $J_+(K \cap \Sigma) \cap J_-(K \cap \Sigma')$ is compact (cf. e. g. Bär *et al.*, 2007, lemma A.5.7). Choose $\Omega \subseteq M$ compact with piecewise smooth boundary $\partial\Omega$, such that the induced metric on the smooth part of $\partial\Omega$ is non-degenerate, and such that $J_+(K \cap \Sigma) \cap J_-(K \cap \Sigma') \subseteq \Omega$ and $K \cap \partial\Omega = K \cap (\Sigma \cup \Sigma')$. (In words, this means: Ω contains the compact set $J_+(K \cap \Sigma) \cap J_-(K \cap \Sigma')$ and lies between Σ and Σ'.)

Let $\mathfrak{n}'$ be the future pointing unit normal vector field along Σ' and let $\tilde{\mathfrak{n}}$ be the outward unit normal vector field along the smooth parts of $\partial\Omega$. Notice that $\tilde{\mathfrak{n}}|_{\partial\Omega \cap K \cap \Sigma} = -\mathfrak{n}|_{\partial\Omega \cap \Sigma}$ and $\tilde{\mathfrak{n}}|_{\partial\Omega \cap K \cap \Sigma'} = \mathfrak{n}'|_{\Omega \cap \Sigma'}$. Then we finally obtain:

$$\int_{\Sigma'} \mathfrak{n}'^a X_a - \int_\Sigma \mathfrak{n}^a X_a = \int_{\partial\Omega} \tilde{\mathfrak{n}}^a \, X_a = \int_\Omega \operatorname{div} X = 0 ,$$

where we used Gauss' divergence theorem for the second equality (in the version for semi-Riemannian manifolds, cf. e. g. Bär *et al.*, 2007, theorem 1.3.16). □

D. General remarks on twisted prenormally hyperbolic operators

Notice that the proof of the previous theorem was not specific to our higher spin construction but can literally be applied to the following general setting:

Theorem 3. Let $(\mathscr{E}, \langle\cdot,\cdot\rangle)$ be a Hermitian vector bundle with a compatible covariant derivative ∇ and let $\mathcal{P}\colon \Gamma(\mathscr{E}) \to \Gamma(\mathscr{E})$ be a prenormally hyperbolic first oder differential operator whose principal symbol $s_{\mathcal{P}}$ is parallel with respect to the induced covariant derivative on $T^*M \otimes \mathrm{End}(\mathscr{E})$, and self-adjoint with respect to $\langle\cdot,\cdot\rangle$, i. e.

$$\forall \xi \in \Gamma(T^*M)\, \forall \Phi, \Psi \in \Gamma(\mathscr{E})\colon \langle \Phi, s_{\mathcal{P}}(\xi)\Psi\rangle = \langle s_{\mathcal{P}}(\xi)\Phi, \Psi\rangle\,.$$

Then for every nowhere vanishing, future pointing, timelike co-vector field $\xi \in \Gamma(T^*M)$, the sesqui-linear form

$$(\Phi, \Psi)_\xi := \langle \Phi, s_{\mathcal{P}}(\xi)\Psi\rangle\,, \qquad \Phi, \Psi \in \Gamma(\mathscr{E})$$

is Hermitian. For a smooth spacelike Cauchy hypersurface $\Sigma \subseteq M$ with future direct unit normal vector field $\mathfrak{n}$, the sesqui-linear form

$$\langle \Phi, \Psi\rangle := \int\limits_\Sigma (\Phi|_\Sigma, \Psi|_\Sigma)_{\mathfrak{n}}$$

on the space of solutions to $\mathcal{P}$ with compactly supported Cauchy data,

$$\mathscr{H} := \{\Phi \in \Gamma(\mathscr{E}) \,|\, \mathcal{P}\Phi = 0 \wedge \mathrm{supp}(\Phi|_\Sigma) \text{ is compact}\}\,,$$

is well-defined (i. e. independent of the choice of Cauchy hypersurface Σ). Moreover, $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ forms a pre-Hilbert space (i. e., $\langle\cdot,\cdot\rangle$ is positive-definite), if $\mathcal{P}$ is of positive-definite with respect to $\langle\cdot,\cdot\rangle$, and $\langle\cdot,\cdot\rangle$ is indefinite if $\mathcal{P}$ is indefinite with respect to $\langle\cdot,\cdot\rangle$.

Proof. For $(\cdot,\cdot)_\xi$ to be Hermitian, cf. remark 5. That the definition of $\mathscr{H}$ does not depend von Σ is a consequence of (Mühlhoff, 2011, corollary 1). The proof that $\langle\cdot,\cdot\rangle$ is well-defined is literally the proof of theorem 2 with $\mathcal{T}$ replaced by $\mathcal{P}$. Notice that compatibility of the covariant derivative on $\mathcal{E}$, parallelness and self-adjointness of $s_{\mathcal{P}}$ are required there. □

In particular, this theorem applies to the following situation of a twisted pre-normally hyperbolic first order operator of positive-definite type:

Lemma 4. Let $\mathcal{P}$ be a first order linear differential operator on the Hermitian vector bundle $(\mathscr{E}, \langle\cdot,\cdot\rangle_{\mathscr{E}})$ which is of positive definite type with respect to $\langle\cdot,\cdot\rangle_{\mathscr{E}}$. Let $(\mathscr{F}, \langle\cdot,\cdot\rangle_{\mathscr{F}})$ be a second Hermitian vector bundle with $\langle\cdot,\cdot\rangle_{\mathscr{F}}$ positive definite. Then the twisted operator $\mathcal{P}^{\mathscr{F}}$ is again of positive definite type with respect to the induced Hermitian scalar product $\langle\cdot,\cdot\rangle_{\mathscr{E}\otimes\mathscr{F}}$ on $\mathscr{E} \otimes \mathscr{F}$.

Proof. Recall that the induced product on $\mathscr{E} \otimes \mathscr{F}$ is on simple tensors $\psi \otimes \alpha$ and $\varphi \otimes \beta \in \Gamma(\mathscr{E} \otimes \mathscr{F})$, $\psi, \varphi \in \Gamma(\mathscr{E})$, $\alpha, \beta \in \Gamma(\mathscr{F})$, given by

$$\langle \psi \otimes \alpha, \varphi \otimes \beta\rangle_{\mathscr{E}\otimes\mathscr{F}} = \langle \psi, \varphi\rangle_{\mathscr{E}} \langle \alpha, \beta\rangle_{\mathscr{F}}\,.$$

The twisted operator $\mathcal{P}^{\mathscr{F}}$ has principal symbol $(s_{\mathcal{P}^{\mathscr{F}}})^a = (s_{\mathcal{P}})^a \otimes \mathrm{Id}_{\mathscr{F}}$ and thus we find for nowhere vanishing, timelike, future pointing $\xi \in \Gamma(T^*M)$:

$$\langle \psi \otimes \alpha, (\xi_a (s_{\mathcal{P}})^a \otimes \mathrm{Id}_{\mathscr{F}})(\varphi \otimes \beta)\rangle_{\mathscr{E}\otimes\mathscr{F}} = \langle \psi, \xi_a (s_{\mathcal{P}})^a \varphi\rangle_{\mathscr{E}} \langle \alpha, \beta\rangle_{\mathscr{F}}\,.$$

This relation extends bilinearly to non-simple tensors and is positive definite as both factors are positive definite by assumption. □

Twisting the Dirac operator by a spinor bundle belonging to an irreducible spinor representation will not lead to a higher spin operator of positive-definite type, as this would require a non-trivial irreducible *unitary* representation of $\mathrm{SL}(2,\mathbb{C})$, which does not exist. This is what we were taken in by in our construction above. However, this does not already dismiss the program of searching for a "nice" set of higher spin field equations, but rather calls for further *conceptual* clarification.

APPENDIX: INTRODUCTION TO $\mathrm{SL}(2,\mathbb{C})$-SPINOR FORMALISM

1. Clifford algebra of Minkowski vector space and its spinor representation

Let $(\mathbb{R}^4, \eta)$ be standard **Minkowski vector space** with $\eta = \mathrm{diag}(1, -1, -1, -1)$ and let $(\mathbb{R}^4, \eta)_\mathbb{C}$ be its complexification (i. e. the complex vector space $\mathbb{C}^4$ equipped with the Hermitian scalar product $\eta_\mathbb{C} = \eta \otimes \mathbb{C}$ obtained by extending η complex anti-linearly in the first and linearly in the second argument). By $\big((\mathbb{R}^4)^*, \eta_\mathbb{C})\big)^*$ we denote the dual space equipped with the induced inverse metric. Let $\mathrm{Cl}^c_{1,3} := \mathrm{Cl}((\mathbb{R}^4, \eta)_\mathbb{C})$ be the complex **Clifford algebra** of complexified Minkowski space (or, which is equivalent, $\mathrm{Cl}^c_{1,3} = \mathrm{Cl}_{1,3} \otimes \mathbb{C}$). It is well known that $\mathrm{Cl}^c_{1,3}$ is isomorphic as $\mathbb{C}$-algebra to $\mathrm{Mat}_{4\times 4}(\mathbb{C})$, the algebra of complex 4×4 matrices. Such an isomorphism $\kappa\colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ represents $\mathrm{Cl}^c_{1,3}$ as matrices and is is called a **spinor representation of $\mathrm{Cl}^c_{1,3}$.** It is an important theorem that all spinor representations of $\mathrm{Cl}^c_{1,3}$ are equivalent.

We call a collection of four complex 4×4 matrices $(\tilde\gamma_0, \dots, \tilde\gamma_3)$ a **collection of Dirac matrices** with respect to an ordered basis $(b_0, \dots, b_3)$ of $\mathbb{R}^4$, if

$$\forall \mu\nu\colon \tilde\gamma_\mu \tilde\gamma_\nu + \tilde\gamma_\nu \tilde\gamma_\mu = 2\, \eta(b_\mu, b_\nu) \cdot \mathbb{1}\,, \tag{1}$$

where $\mathbb{1}$ denotes the 4×4 identity matrix. Collections of Dirac matrices are intimately related to spinor representations of $\mathrm{Cl}^c_{1,3}$: From a spinor representation $\kappa\colon \mathrm{Cl}^c_{1,3} \to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ we obtain a collection of Dirac matrices with respect to a chosen basis $(b_0, \dots, b_3)$ by setting $\tilde\gamma_\mu := \kappa(b_\mu)$. Vice versa, if a collection of Dirac matrices $(\tilde\gamma_0, \dots, \tilde\gamma_3)$ with respect to a basis $(b_0, \dots, b_3)$ is given, we obtain a spinor representation κ of $\mathrm{Cl}^c_{1,3}$ by setting

$$\kappa(x) := x^\mu \tilde\gamma_\mu \quad \text{for vectors } x = x^\mu b_\mu \in \mathbb{R} \subseteq \mathrm{Cl}^c_{1,3}\,.$$

This is to be multiplicatively extended to all of $\mathrm{Cl}^c_{1,3}$ and it follows from the defining relations (1) that this yields a well-defined representation of $\mathrm{Cl}^c_{1,3}$.

On the level of Dirac matrices, equivalence of all spinor representations of $\mathrm{Cl}^c_{1,3}$ is equivalent to the following two statements about **basis transformations,** which are often useful: If $b_\mu \mapsto b'_\mu = A^\nu{}_\mu b_\nu$ for an invertible matrix $A^\nu{}_\mu \in \mathrm{GL}(4, \mathbb{C})$ is a basis transformation in $(\mathbb{R}^4, \eta)$, then the matrices $\tilde\gamma'_\mu := A^\nu{}_\mu \tilde\gamma_\nu$ form a collection of Dirac matrices with respect to the transformed basis $(b'_0, \dots, b'_3)$, if $(\tilde\gamma_0, \dots, \tilde\gamma_3)$ is a collection of Dirac matrices with respect to $(b_0, \dots, b_3)$, as can be checked easily. (I. e., the $\tilde\gamma_\mu$ **transform covariantly** under a basis transformation in $(\mathbb{R}^4, \eta)_\mathbb{C}$, and this is why the index μ is written as subscript index.) Moreover, according to a famous theorem by Pauli, if both $(\tilde\gamma_0, \dots, \tilde\gamma_3)$ and $(\tilde\gamma'_0, \dots, \tilde\gamma'_3)$ are collections of Dirac matrices with respect to a basis $(b_0, \dots, b_3)$, there is an invertible 4×4 matrix S, so that $\tilde\gamma'_\mu = S \tilde\gamma_\mu S^{-1}$ (basis transformation in the representation space). This matrix S is just the intertwining isomorphism of the two spinor representations of $\mathrm{Cl}^c_{1,3}$ induced by the two different collections of Dirac matrices.

This shows that **making a choice of Dirac matrices** in combination with a choice of bases of $(\mathbb{R}^4, \eta)$ comprises nothing more than the fixation of one out of all the equivalent spinor representations of $\mathrm{Cl}^c_{1,3}$. Though all of our formulas will make use of an invariant notation, we will now choose a certain collection of Dirac matrices and bases which is particularly convenient for easily switching between Dirac 4-spinor notation and 2-spinor notation. We are free to make this choice, our theory doesn't depend on it, but we gain notational elegance.

First, we declare the well-known 2×2 **Pauli spin matrices** (extended by $\tilde\sigma_0$):

$$\tilde\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Recall that the following relations hold (as always, Latin indices i, j, k take values $1, 2, 3$, Greek indices μ, ν take values $0, \dots, 3$):

$$[\tilde\sigma_i, \tilde\sigma_j] = 2i\, \epsilon_{ijk}\, \tilde\sigma_k\,, \qquad \tilde\sigma_\mu^2 = \tilde\sigma_0\,, \qquad \tilde\sigma_\mu^\dagger = \tilde\sigma_\mu^{-1} = \tilde\sigma_\mu\,.$$

Then it is easily checked that the following forms a collection of Dirac matrices with respect to the **standard basis** $(e_0, \dots, e_3)$ **of** $(\mathbb{R}^4, \eta)_{\mathbb{C}}$**:**

$$\tilde{\gamma}_0 := \begin{pmatrix} 0 & \tilde{\sigma}_0 \\ \tilde{\sigma}_0 & 0 \end{pmatrix}, \qquad \tilde{\gamma}_i := \begin{pmatrix} 0 & \tilde{\sigma}_i \\ -\tilde{\sigma}_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3\,.$$

The spinor representation of $\mathrm{Cl}^c_{1,3}$ induced by this collection of Dirac matrices is commonly referred to as **Weyl representation.** From this point on, we will always be using this representation, as performing concrete calculations requires a concrete fixation of one out of all the equivalent spinor representations of $\mathrm{Cl}^c_{1,3}$. However, none of our results depends on this choice, this choice just leads to notational benefits (which comes from the fact that the Weyl representation already encodes a direct sum decomposition of the Dirac spinor representation of $\mathrm{Spin}_0(1,3)$ into its chiral parts, which enables easy switching between 4-spinor and 2-spinor notation, see below).

2. The Spin-group of Minkowski vector space and $\mathrm{SL}(2,\mathbb{C})$

Recall that every Clifford algebra $\mathrm{Cl}(V)$ for a real or complex vector space V carries a natural $\mathbb{Z}_2$ grading, this is a direct sum decomposition $\mathrm{Cl}(V) = \mathrm{Cl}^+(V) \oplus \mathrm{Cl}^-(V)$, where $\mathrm{Cl}^{\pm}(V)$ are the ± 1 eigenspaces of the grading automorphism $\alpha\colon \mathrm{Cl}(V) \to \mathrm{Cl}(V)$, which is given by multiplicative and additive extension of $\alpha(v) := -v$ for $v \in V$. The Spin group of the Clifford algebra $\mathrm{Cl}^c_{1,3}$ is then defined as $\mathrm{Spin}(1,3) := \mathrm{Pin}(1,3) \cap (\mathrm{Cl}^c_{1,3})^+$, where $\mathrm{Pin}(1,3) \subseteq \mathrm{Cl}^c_{1,3}$ is the multiplicative subgroup of $\mathrm{Cl}^c_{1,3}$ generated by the subset $\{x \in \mathbb{C}^4 \,|\, \eta_{\mathbb{C}}(x,x) = \pm 1\}$. It is well known that $\mathrm{Spin}(1,3)$ is a *real* 6-dimensional Lie group, and it is generated by the bi-vectors

$$M_i := \epsilon_{ijk}\, e_j e_k\,, \quad N_i := e_i e_0\,, \quad i,j,k = 1,2,3\,.$$

(This means that the M_i, N_i form a basis of the real Lie-algebra $\mathfrak{spin}(1,3)$ of $\mathrm{Spin}_0(1,3)$.) Here, $(e_0, \dots, e_3)$ denotes the standard basis of $(\mathbb{R}^4, \eta)_{\mathbb{C}} \subseteq \mathrm{Cl}^c_{1,3}$. It is easily checked that amongst these generators, the following commutator relations hold:

$$[M_i, M_j] = 2\,\epsilon_{ijk}\, M_k\,, \quad [N_i, N_j] = -2\,\epsilon_{ijk}\, M_k\,, \quad [M_i, N_j] = 2\,\epsilon_{ijk}\, N_k\,. \tag{CR}$$

The central role of $\mathrm{Spin}(1,3)$ for our considerations is due to the fact that it is the natural universal covering group of the special orthogonal group $\mathrm{SO}(1,3) = \mathcal{L}_+$, where $\mathcal{L}_+$ denotes the group of Lorentz transformations with positive determinant. $\mathrm{Spin}(1,3)$ has two connected components and we denote the connected component of unity by $\mathrm{Spin}_0(1,3)$. It universally covers the proper orthochronous Lorentz group $\mathcal{L}^{\uparrow}_+$ (Lorentz transformation without space or time reversal).

Finally, it is well known that $\mathrm{Spin}_0(1,3)$ is as real Lie group isomorphic to $\mathrm{SL}(2,\mathbb{C})$, the group of unimodular complex 2×2 matrices. As will become apparent below, spinor calculations are easier when dealing with elements of $\mathrm{SL}(2,\mathbb{C})$ (i. e., matrices) instead of elements of $\mathrm{Spin}_0(1,3)$, and therefore we will now fix a concrete isomorphism $\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Spin}_0(1,3)$. For $\mathrm{SL}(2,\mathbb{C})$ we have a set of generators given by

$$m_i := -i\,\tilde{\sigma}_i\,, \quad n_i := \tilde{\sigma}_i\,, \quad i = 1,2,3\,.$$

It is easily checked that they fulfill the same commutator relations as (CR) with capital letters replaced by lower case letters. Thus we obtain a Lie algebra isomorphism $d\varphi\colon \mathfrak{spin}(1,3) \to \mathfrak{sl}(2,\mathbb{C})$ (we denote the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ by $\mathfrak{sl}(2,\mathbb{C})$) by linear extension of

$$\begin{aligned} d\varphi\colon m_i &\mapsto M_i\,, \qquad i = 1,2,3 \\ n_i &\mapsto N_i\,. \end{aligned}$$

As $\mathrm{Spin}_0(1,3)$ and $\mathrm{SL}(2,\mathbb{C})$ are simply connected, this integrates uniquely to a Lie group isomorphism $\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{Spin}_0(1,3)$.

3. The Dirac spinor representation of $\mathrm{SL}(2,\mathbb{C})$ and its chiral decomposition

By $\mathrm{Mat}_{k\times k}(\mathbb{C})$ we denote the algebra of complex $k\times k$ matrices. By $\mathrm{GL}(k,\mathbb{C})$ we denote the subgroup of invertible $k\times k$ matrices. It is a Lie group with Lie algebra $\mathfrak{mat}_{4\times 4}(\mathbb{C})$, the space of complex 4×4-matrices with matrix commutator.

The restriction

$$D^D := \kappa|_{\mathrm{Spin}_0(1,3)}\colon \mathrm{Spin}_0(1,3)\to \mathrm{GL}(4,\mathbb{C})$$

of a spinor representation $\kappa\colon \mathrm{Cl}^c_{1,3}\to \mathrm{Mat}_{4\times 4}(\mathbb{C})$ to the group $\mathrm{Spin}_0(1,3)\subseteq \mathrm{Cl}^c_{1,3}$ is called **Dirac spinor representation of** $\mathrm{Spin}_0(1,3)$**.** As all spinor representations of $\mathrm{Cl}^c_{1,3}$ are equivalent, a Dirac spinor representation of $\mathrm{Spin}_0(1,3)$ is unique up to isomorphism. Let $dD^D\colon \mathfrak{spin}(1,3)\to \mathfrak{mat}_{4\times 4}(\mathbb{C})$ denote the associated infinitesimal representation of the Lie algebra $\mathfrak{spin}(1,3)$. Then with respect to our concretely chosen spinor representation κ (Weyl representation), we find for the generators M_i and N_i of $\mathfrak{spin}(1,3)$:

$$dD^D(M_i)=\epsilon_{ijk}\,\tilde\gamma_j\tilde\gamma_k=\begin{pmatrix}-i\,\tilde\sigma_i & 0\\ 0 & -i\,\tilde\sigma_i\end{pmatrix},\qquad dD^D(N_i)=\tilde\gamma_i\tilde\gamma_0=\begin{pmatrix}\tilde\sigma_i & 0\\ 0 & -\tilde\sigma_i\end{pmatrix}. \tag{GD}$$

All six generators of D^D are of block diagonal form. Hence, the Dirac spinor representation **splits into a direct sum** of two 2-dimensional **chiral subrepresentations.** For the first chiral subrepresentation, $D^+\colon \mathrm{Spin}_0(1,3)\to \mathrm{GL}(2,\mathbb{C})$, we take the upper left block of equations (GD), which means that, on the level of generators, $dD^+(M_i)=-i\,\tilde\sigma_i$ and $dD^+(N_i)=\tilde\sigma_i$. Notice that by concatenation with the isomorphism $\varphi\colon \mathrm{SL}(2,\mathbb{C})\to \mathrm{Spin}_0(1,3)$ defined above, we obtain a representation $D^+\circ\varphi$ of $\mathrm{SL}(2,\mathbb{C})$, infinitesimally given by

$$\begin{aligned} d(D^+\circ\varphi)(m_i)&=(dD^+\circ d\varphi)(m_i)=-i\,\tilde\sigma_i=m_i\,,\\ d(D^+\circ\varphi)(n_i)&=(dD^+\circ d\varphi)(n_i)=\tilde\sigma_i=n_i\,.\end{aligned}$$

In words: The generators $m_i,n_i\in\mathfrak{sl}(2,\mathbb{C})\subseteq\mathfrak{mat}_{2\times 2}(\mathbb{C})$ get represented by themselves. This shows that $D^+\circ\varphi$ **is the defining representation of** $\mathrm{SL}(2,\mathbb{C})$**,** i. e. $(D^+\circ\varphi)(S)=S$ for $S\in\mathrm{SL}(2,\mathbb{C})\subseteq\mathrm{GL}(2,\mathbb{C})$. This is the reason why it is much more handy taking $\mathrm{SL}(2,\mathbb{C})$ instead of $\mathrm{Spin}_0(1,3)$ when doing calculations with 2-spinors. We shall often drop the $\circ\varphi$ and just write $D^+(S)$ and $D^D(S)$ for $S\in\mathrm{SL}(2,\mathbb{C})$.

Recall that for a general representation of a Lie group G, $\rho\colon G\to\mathrm{GL}(k,\mathbb{C})$, the **complex conjugate representation** $\bar\rho$ is given by $\bar\rho(g):=\overline{\rho(g)}$, $g\in G$, and the **dual representation** (contragredient representation), ρ^*, on the dual space of the representation space, is given by $\rho^*(g):=(\rho(g^{-1}))^{\mathrm{tr}}$ (here, the superscript $^{\mathrm{tr}}$ means the transpose matrix). On the Lie algebra level, this means for $\xi\in\mathfrak{g}$: $d(\rho^*)(\xi)=-(d\rho(\xi))^{\mathrm{tr}}$. Now, combining both, we consider $\overline{D^+}^*$, and find for the infinitesimal representation $d(\overline{D^+}^*)\colon\mathfrak{spin}(1,3)\to\mathfrak{mat}_{2\times 2}(\mathbb{C})$:

$$\begin{aligned} d(\overline{D^+}^*)(M_i)&=-\overline{d(D^+)(M_i)}^{\mathrm{tr}}=-\overline{(-i\,\tilde\sigma_i)}^{\mathrm{tr}}=-i\,\tilde\sigma_i^\dagger=-i\,\tilde\sigma_i\,,\\ d(\overline{D^+}^*)(N_i)&=-\overline{d(D^+)(N_i)}^{\mathrm{tr}}=-\overline{\tilde\sigma_i}^{\mathrm{tr}}=-\tilde\sigma_i^\dagger=-\tilde\sigma_i\,.\end{aligned}$$

Comparing this with the lower right blocks of equations (GD) and writing $D^-:=\overline{D^+}$, this finally shows for the Dirac spinor representation $D^D=\kappa|_{\mathrm{Spin}_0(1,3)}$:

$$D^D=D^+\oplus(D^-)^*\,.$$

This is the chiral decomposition of D^D, and the $D^\pm$ are called **positive resp. negative Weyl spinor representation.** As shown above, as representations of $\mathrm{SL}(2,\mathbb{C})$, $D^+(S)=S$, for $S\in\mathrm{SL}(2,\mathbb{C})$, is the defining representation, $D^-(S)=\bar S$ is the complex conjugate of the defining representation, and $(D^-)^*(S)=(\bar S^{-1})^{\mathrm{tr}}$. Considering the second summand in the chiral decomposition of D^D as dual representation makes it possible to have D^+ and D^- a complex conjugate pair of representations, which will lead to an elegant notation of 2-spinors below (by the way, the representations D^- and $(D^-)^*$ are equivalent, as is easily seen using the totally anti-symmetric ϵ-spinor introduced below).

4. Classification of finite dimensional $\mathrm{SL}(2,\mathbb{C})$ spinor representations

If $D\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{GL}(V)$ is a finite dimensional representation of $\mathrm{SL}(2,\mathbb{C}) \cong \mathrm{Spin}_0(1,3)$ on a complex vector space V, we call the elements of V **spinors of type** D**.** In particular, the spinors of type D^D are called **Dirac spinors** or **4-spinors** (as D^D is 4-dimensional), and spinors of types $D_\pm$ are called **positive/negative Weyl spinors.** Lets denote the representation space of D^D by Δ_D (of course, $\Delta_D = \mathbb{C}^4$) and the representation spaces of $D^\pm$ by $\Delta_\pm$ ($\Delta_\pm = \mathbb{C}^2$). As $D^D = D^+ \oplus (D^-)^*$, we have $\Delta_D = \Delta_+ \oplus \Delta_-^*$.

It turns out that the two Weyl representation, D^+ and D^-, are not equivalent and that they are (up to equivalence) the only irreducible 2-dimensional representations of $\mathrm{SL}(2,\mathbb{C})$. Moreover, they form a set of **fundamental representations of** $\mathrm{SL}(2,\mathbb{C})$**,** which means that every complex finite dimensional **irreducible representation of** $\mathrm{SL}(2,\mathbb{C})$ **is isomorphic to a symmetrized tensor product of** D^+ **and** D^-**,** i. e. to a representation of the form

$$D^{(\frac{k}{2},\frac{l}{2})} := (D^+)^{\vee k} \otimes (D^-)^{\vee l} \quad \text{on} \quad \Delta_{\frac{k}{2},\frac{l}{2}} := (\Delta_+)^{\vee k} \otimes (\Delta_-)^{\vee l}\,, \quad k,l \in \mathbb{N}\,,$$

were $X^{\vee k}$ denotes the k-fold symmetrized tensor product $X^{\vee k} := X \vee \ldots \vee X$ (k times). The number $\frac{k}{2} + \frac{l}{2}$ will be interpreted as the physical particle spin. For the dimensions of these representations it is easily checked that

$$\dim D^{(\frac{k}{2},\frac{l}{2})} = (2k+1)(2l+1)\,.$$

Notice that in this systematic notation, $D^+ = D^{(\frac{1}{2},0)}$ and $D^- = D^{(0,\frac{1}{2})}$. Moreover, as $D^- = \bar{D}^+$, we find that complex conjugation just switches k and l:

$$\overline{D^{(\frac{k}{2},\frac{l}{2})}} \cong D^{(\frac{l}{2},\frac{k}{2})} \quad \text{on} \quad \overline{\Delta_{\frac{k}{2},\frac{l}{2}}} \cong \Delta_{\frac{l}{2},\frac{k}{2}}\,.$$

It is important to bear in mind that only fully symmetrized tensor products of irreducible spinor representations are again irreducible. This can also be seen from the **Clebsch-Gordon formula,** which reads in our setting:

$$D^{(\frac{k}{2},\frac{l}{2})} \otimes D^{(\frac{k'}{2},\frac{l'}{2})} \cong \bigoplus_{i=|k-k'|}^{k+k'} \bigoplus_{j=|l-l'|}^{l+l'} D^{(\frac{i}{2},\frac{j}{2})}\,.$$

As an important special case, we have:

$$D^{(\frac{1}{2},0)} \otimes D^{(\frac{k}{2},\frac{l}{2})} \cong D^{(\frac{k+1}{2},\frac{l}{2})} \oplus D^{(\frac{k-1}{2},\frac{l}{2})}\,.$$

5. Index notation for 2-spinors

Though they are both 2-dimensional complex vector spaces, it is important to formally distinguish the representation spaces $\Delta_{\frac{1}{2},0}$ and $\Delta_{0\frac{1}{2}}$. This is why in index notation we use **undotted capital indices** A, B, ..., X, Y, ..., for positive, and **dotted capital indices** $\dot{A}$, $\dot{B}$, ..., $\dot{X}$, $\dot{Y}$, ... for negative Weyl spinors:

$$\psi = \psi^A \in \Delta_{\frac{1}{2},0} \quad \text{and} \quad \varphi = \varphi^{\dot{X}} \in \Delta_{0,\frac{1}{2}}\,.$$

As usual, co-spinor indices are subscripted:

$$\psi = \psi_A \in \Delta^*_{\frac{1}{2},0} \quad \text{and} \quad \varphi = \varphi_{\dot{X}} \in \Delta^*_{0,\frac{1}{2}}\,.$$

A mixed (higher) spinor $\Psi \in \Delta_{\frac{k}{2},\frac{l}{2}}$ will then be denoted

$$\Psi = \Psi^{A_1\ldots A_k\dot{X}_1\ldots\dot{X}_l} \in \Delta_{\frac{k}{2},\frac{l}{2}}\,.$$

All these indices are considered **abstract indices,** i. e. they are *not* numerical variables taking values (for such we would use Greek letters μ, ν, ...), rather, they are abstract labels.

The main benefit from abstract index notation is that it maintains the formal aspects of the Einstein summation convention and of index shifting (isomorphic mapping between vector space and its dual), but still **all expressions are invariant,** they do not assume a choice of reference basis.[3]

The transformation of spinors under an $S \in \mathrm{SL}(2,\mathbb{C})$ reads in this notation:

$$D^{(\frac{1}{2},0)}(S)\psi = S^A{}_B\,\psi^B\,, \quad D^{(0,\frac{1}{2})}(S)\varphi = \bar{S}^{\dot{X}}{}_{\dot{Y}}\,\varphi^{\dot{Y}}\,,$$
$$D^{(\frac{k}{2},\frac{l}{2})}(S)\Psi = S^{A_1}{}_{B_1}\cdots S^{A_k}{}_{B_k}\,\bar{S}^{\dot{X}_1}{}_{\dot{Y}_1}\cdots\bar{S}^{\dot{X}_l}{}_{\dot{Y}_l}\,\Psi^{B_1\dots B_k\dot{Y}_1\dots\dot{Y}_l}\,.$$

We are using **implicit contraction** (analog to implicit summation in classical component index notation), but this may only be taken across indices which are either both dotted or both undotted, and only diagonally (one index upper, the other lower, i. e. one spinor, the other co-spinor).

Once again, notice that spinors in the irreducible representation $\Delta_{\frac{k}{2},\frac{l}{2}}$ are fully symmetric. This means that a $\Psi \in \Delta_{\frac{1}{2},0}^{\otimes k}\otimes\Delta_{0,\frac{1}{2}}^{\otimes l}$ is an element of the *smaller* space $\Delta_{\frac{k}{2},\frac{l}{2}}$, if and only if

$$\Psi^{A_1\dots A_k\dot{X}_1\dots\dot{X}_l} = \Psi^{(A_1\dots A_k)(X_1\dots\dot{X}_l)}\,.$$

Here, the brackets are the **symmetrization operator:**

$$\Phi^{(A_1\dots A_k)} := \frac{1}{k!}\sum_{\pi\in S_k}\Phi^{A_{\pi(1)}\dots A_{\pi(k)}}\,,$$

where S_k is the group of permutations of numbers $1,\dots,k$. This definition holds analogously for symmetrization on dotted indices.

As it is a 2-dimensional space, $\Delta_{\frac{1}{2},0} = \Delta_+ = \mathbb{C}^2$ carries a unique (up to a scalar factor) symplectic structure ε_{AB}, i. e. an anti-symmetric bilinear form. We fix this form by declaring that with respect to the standard basis of $\mathbb{C}^2$ it has the component representation

$$\varepsilon_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We call this **the ε-spinor.** On the co-spinor space Δ_+^*, we declare ε^{AB} such that $\varepsilon^{AB}\varepsilon_{CB} = \mathrm{Id}^A{}_C$, where Id denotes the identity matrix. By complex conjugation we obtain $\varepsilon_{\dot{X}\dot{Y}}$ on $\Delta_{0,\frac{1}{2}} = \Delta_-$ and $\varepsilon^{\dot{X}\dot{Y}}$. It is easily checked that ε_{AB} is invariant under $D^{\frac{1}{2},0}\otimes D^{\frac{1}{2},0}$ (analogously for ε^{AB}, $\varepsilon_{\dot{X}\dot{Y}}$ and $\varepsilon^{\dot{X}\dot{Y}}$.

Notice, we didn't designate a scalar product on Δ_+ and Δ_- with respect to which we might shift indices (i. e. use the induced isomorphism between space and dual space). **Index shifting** in 2-spinor formalism though will be done with respect to the anti-symmetric ε-spinors: **Lowering with the first, raising with the second index of ε.** We have:

$$\psi_B = \varepsilon_{AB}\,\psi^A\,, \qquad \varphi_{\dot{Y}} = \varepsilon_{\dot{X}\dot{Y}}\,\varphi^{\dot{X}}$$
$$\psi^A = \varepsilon^{AB}\,\psi_B\,, \qquad \varphi^{\dot{X}} = \varepsilon_{\dot{X}\dot{Y}}\,\varphi_{\dot{Y}}$$

6. σ-tensor spinor and vector representation

We declare **the σ-tensor spinor,** which is a mixed object $\sigma_a{}^{A\dot{X}} \in (\mathbb{R}^4_{\mathbb{C}})^*\otimes\Delta_{\frac{1}{2},\frac{1}{2}}$, where $\mathbb{R}^4_{\mathbb{C}} = \mathbb{C}^4$ denotes complexified Minkowski vector space. With respect to the standard bases in $(\mathbb{R}^4,\eta)_{\mathbb{C}}$ and $\Delta_\pm$, it is given by the Pauli matrices:

$$[\sigma_\mu{}^{\kappa\dot{\lambda}}] := \frac{1}{\sqrt{2}}\,\tilde{\sigma}_\mu\,, \qquad \mu = 0,\dots,3\,.$$

[3] Notice: An expression holds in abstract index notation if and only if *for every choice of reference basis,* it holds in component index notation. Thus, also the reader not familiar with abstract index notation can follow this document if he just assumes his favorite choice of basis (or, later, local reference frame) chosen.

(κ = row index, $\dot\lambda$ = column index). ${\sigma_a}^{A\dot X}$ can be considered as a map $\sigma\colon (\mathbb{R}^4,\eta)_{\mathbb{C}} \to \Delta_{\frac12,\frac12}$, given for $x = x^a \in \mathbb{R}^4_{\mathbb{C}}$ by

$$\sigma(x) = x^a {\sigma_a}^{A\dot X} =: x^{A\dot X}\,.$$

It is easy to see that σ is an isomorphism and that the inverse $\sigma^{-1}\colon \Delta_{\frac12,\frac12} \to (\mathbb{R}^4,\eta)_{\mathbb{C}}$ is given by ${\sigma^a}_{A\dot X}$, where all indices are shifted according to the rules (i. e., using ε_{AB}, $\varepsilon_{\dot X\dot Y}$ and η^{ab}).

The σ-spinor plays a central role in the formalism because it can be shown that for every $S \in \mathrm{SL}(2,\mathbb{C})$, the mapping $\lambda(S)\colon (\mathbb{R}^4,\eta)_{\mathbb{C}} \to (\mathbb{R}^4,\eta)_{\mathbb{C}}$, defined by the diagram

$$\begin{tikzcd}
(\mathbb{R}^4,\eta)_{\mathbb{C}} \ni x^a \arrow[r, mapsto, "\lambda(S)"] \arrow[d, mapsto, "\sigma"] & \lambda(S)x \in (\mathbb{R}^4,\eta)_{\mathbb{C}} \arrow[d, mapsto, "\sigma"] \\
\Delta_{\frac12,\frac12} \ni x^{A\dot X} \arrow[r, mapsto, "D^{(\frac12,\frac12)}(S)"] & {S^A}_B\, {\bar S^{\dot X}}_{\dot Y}\, x^{B\dot Y} \in \Delta_{\frac12,\frac12}
\end{tikzcd}$$

is a restricted, orthochronous Lorentz transformation. λ as defined by commutativity of this diagram is the **universal covering map** $\lambda\colon \mathrm{SL}(2,\mathbb{C}) \to \mathcal{L}_+^\uparrow$ and, as the commutative diagram tells us, the representation of $\mathrm{SL}(2,\mathbb{C})$ through λ on $(\mathbb{R}^4,\eta)_{\mathbb{C}}$ is *equivalent* to $D^{(\frac12,\frac12)}$ on $\Delta_{\frac12,\frac12}$, with σ as intertwiner!

This is the reason why the $\mathrm{SL}(2,\mathbb{C})$-representation $D^{(\frac12,\frac12)}$ is also referred to as **vector representation** (notice, it has spin 1). Every vector $x = x^a \in (\mathbb{R}^4,\eta)_{\mathbb{C}}$ naturally corresponds to a $D^{(\frac12,\frac12)}$-spinor

$$x^{A\dot X} := x^a\, {\sigma_a}^{A\dot X}\,.$$

The internal consistency of this formalism gets apparent one more time when we consider the push-forward of η_{ab} through σ:

$$\eta_{A\dot X B\dot Y} = \eta_{ab}\, {\sigma^a}_{A\dot X}\, {\sigma^b}_{B\dot Y}\,.$$

An easy calculation reveals:

$$\eta_{A\dot X B\dot Y} = \varepsilon_{AB}\varepsilon_{\dot X\dot Y}\,.$$

This means: The canonical *anti-symmetric* structures ε on $\Delta_{\frac12,0}$ and $\Delta_{0,\frac12}$ multiply together to form the *symmetric* Lorentzian scalar product $\eta_{\mathbb{C}}$ on $(\mathbb{R}^4,\eta)_{\mathbb{C}}$.

7. The γ-tensor spinor

Recall that the spinor representation κ represents $\mathrm{Cl}^c_{1,3}$ as matrices on $\mathbb{C}^4$. This representation was fixed by our choice of Dirac matrices $\tilde\gamma_\mu$. Restricting κ to $\mathrm{Spin}_0(1,3)$, we called the representation space $\mathbb{C}^4 =: \Delta_D$ and introduced the chiral decomposition $\Delta_D = \Delta_{\frac12,0} \oplus \Delta^*_{0,\frac12}$. Let a Dirac spinor $\Psi \in \Delta_D$ be given with respect to this chiral decomposition as

$$\Psi = \begin{pmatrix} (\psi_1)^A \\ (\psi_2)_{\dot X} \end{pmatrix}, \qquad \psi_1 = (\psi_1)^A \in \Delta_{\frac12,0}\,, \quad \psi_2 = (\psi_2)_{\dot X} \in \Delta^*_{0,\frac12}\,.$$

Let a vector $x = x^a \in (\mathbb{R}^4,\eta)_{\mathbb{C}} \subseteq \mathrm{Cl}^c_{1,3}$ be given in components x^μ with respect to the standard basis $(e_0,\dots,e_3)$. Then we find by application of all the definitions we made:

$$\kappa(x)\Psi = x^\mu\,\tilde\gamma_\mu\Psi = \begin{pmatrix} 0 & x^\mu\,\tilde\sigma_\mu \\ \eta_{\mu\mu}\,x^\mu\,\tilde\sigma_\mu & \end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \sqrt2\begin{pmatrix} x^a\,{\sigma_a}^{A\dot X}\,(\psi_2)_{\dot X} \\ x^a\,\sigma_{a\,\dot XA}\,(\psi_1)^A\end{pmatrix}.$$

Thus, the Dirac matrices $\tilde{\gamma}_\mu$ with respect to the standard basis $(e_0, \ldots, e_3)$ give rise to the so called **γ-tensor spinor** $\gamma_a \in (\mathbb{R}^4, \eta)^*_{\mathbb{C}} \otimes \mathrm{GL}(\Delta_D)$, which might as well be considered a map $\gamma\colon (\mathbb{R}^4, \eta)_{\mathbb{C}} \to \mathrm{GL}(\Delta_D)$. Accounting for the chiral decomposition $\Delta_D = \Delta_{\frac{1}{2},0} \oplus \Delta^*_{0,\frac{1}{2}}$, our choice of Dirac matrices (the Weyl representation, cf. section 1) induces γ_a to be given by:

$$\gamma_a = \begin{pmatrix} 0 & {\sigma_a}^{A\dot{X}} \\ \sigma_{a\,\dot{X}A} & 0 \end{pmatrix}.$$

(Notice, in this notation, the sign change of $\eta_{\mu\mu}\,\tilde{\sigma}_\mu$ for $\mu = 1, 2, 3$ in the lower left corner is implemented implicitly when the indices $A\dot{X}$ are shifted down by means of $\varepsilon_{AB}\varepsilon_{\dot{X}\dot{Y}}$.) This is the reason we prefer the Weyl representation. It allows easy transition between Dirac spinor notation and 2-spinor notation.

8. Simultaneous reference frame transformation and invariance of ε, σ and γ

"2-spinors are objects that change sign under a rotation by 2π"—this is what is often written in text books. It just says that in general, a Lorentz transformation does not contain all the information about a **reference frame transformation** of a physical system which is described by elements of the Minkowski vector space $(\mathbb{R}^4, \eta)$ and spinors of type D, where D is a general finite-dimensional representation of $\mathrm{SL}(2, \mathbb{C})$ on a complex vector space Δ. Rather, such a transformation is fully qualified only by giving an element of $\mathrm{SL}(2, \mathbb{C})$:

Let a reference frame of our system by given by bases $(b_0, \ldots, b_3)$ of $(\mathbb{R}^4, \eta)$ and $(B_1, \ldots, B_k)$ of Δ $(k := \dim \Delta)$. Moreover, let $\lambda\colon \mathrm{SL}(2, \mathbb{C}) \to \mathcal{L}_+^\uparrow$ denote the universal covering map as given above. Then an element $S \in \mathrm{SL}(2, \mathbb{C})$ induces a reference frame transformation

$$b_\mu \mapsto b'_\mu := \lambda(S)(b_\mu) \quad \text{and} \quad B_i \mapsto B'_i := D(S)(B_i)\,, \quad i = 1, 2\,.$$

Now, as a very useful property of the formalism it can be shown that the objects ε^{AB}, ${\sigma_a}^{A\dot{X}}$ and γ_a are **invariant** under such simultaneous reference frame transformations! For example in the case of ${\sigma_a}^{A\dot{X}}$, this means that the component representation of ${\sigma_a}^{A\dot{X}}$ with respect to bases $(b^0, \ldots, b^3)$ of $(\mathbb{R}^4, \eta)^*_{\mathbb{C}}$, (B_1, B_2) of $\Delta_{\frac{1}{2},0}$ and $(\bar{C}_1, \bar{C}_2)$ of $\Delta_{0,\frac{1}{2}}$ remains untouched when these bases are simultaneously transformed under $\lambda(S)^*$, $D^{(\frac{1}{2},0)}(S)$ and $D^{(0,\frac{1}{2})}(S)$ for any $S \in \mathrm{SL}(2, \mathbb{C})$. Analog statements hold for ε^{AB}, ε_{AB}, $\varepsilon^{\dot{X}\dot{Y}}$, $\varepsilon_{\dot{X}\dot{Y}}$ and γ_a.

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